Logistical growth, chaotical behaviour and fishing quota.

Content

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1 Introduction.

The theory of logistical growth and all kinds of dynamical processes are well described in textbooks (Strogatz, S.H. ; Railsback, S.F and V. Grimm). Non-linear dynamics, complexity and chaos are most interesting in disrupting the established theoretical models of e.g. economics. Interesting reading about the latter subject is the book of Waldrop. In the column Free Exchange of The Economist mentioned: “the teaching of economics gets an overdue overhaul”. Hopefully, complexity and non-linear dynamics is paid attention to in the curriculum of universities.

Here we will look into a simple model. A first order non-linear differential equation. This equation describes for example population growth(The Lotka- Volterra model) and decline due to fishing quota.

2 The model equation.

The equations reads:

$$\frac{dF}{dt} = g(F-d)(1-F) - cF. \qquad (1)$$

In this equation:

- F represents the “normalized” population. This normalisation is done with the carrying capacity.
- g is a growth factor; a constant and >0
- d is a threshold below which the reproduction is too low for a sustainable population; a constant and >0
- c is the catch factor or quota. It could also represent predation; a constant and >0. It would be rather curious a fisherperson bringing her/his catch back to the sea.

- $t \geq 0$ is time.
Strogatz discussed this type of equation by means of a geometrical way of thinking (Strogatz Ch. 2).
We will discuss the exact solution.

3 The solution.

We rewrite Eq. (1):

$$\frac{1}{g}\frac{dF}{dt} = (F-d)(1-F) - aF, \quad (2)$$

and $a = \frac{c}{g}$ and denominate a the catch factor.
Eq.(2) is solved by means of separation and factorization:

$$F = \frac{A + CBe^{-(A-B)gt}}{1 + Ce^{-(A-B)gt}}, \quad (3)$$

and

$$A = \frac{1}{2}[(1+d-a) + \sqrt{(1+d-a)^2 - 4d}], \quad (4)$$

$$B = \frac{1}{2}[(1+d-a) - \sqrt{(1+d-a)^2 - 4d}]. \quad (5)$$

So $A > B$, except for the case $\sqrt{(1+d-a)^2 - 4d} = 0$.
C represent a constant of integration and is determined by the initial condition $F(t=0) = F_0$:

$$C = \frac{A - F_0}{F_0 - B}. \quad (6)$$

Eqs. (4) , (5) and (6) become complex numbers for $4d > (1+d-a)^2$, or

$$(1 + d - 2\sqrt{d}) < a < (1 + d + 2\sqrt{d}). \quad (7)$$

In the next section we will discuss in 3-D space various values of a and $F(a,t)$ as a function of time.

4 Discussion.

In the foregoing section we mentioned the possibility of complex numbers. Hence, does Eq.(7) represent a constraint? Let us find out.

4.1 $(\mathbf{1} + \boldsymbol{d} - \mathbf{2}\sqrt{\boldsymbol{d}}) > \boldsymbol{a} > \mathbf{0}$.

Let us start with $(1 + d - 2\sqrt{d}) > a$ and Eq. (3).
For $B < F_0 < A$ we find a smooth logistic curve, starting at $F(t=0) = F_0$ and for $\lim_{t\to\infty} F = A$.
Linear stability analysis (Strogatz) also shows $(1 + d - 2\sqrt{d}) > a$ to be stable .
The logistic curve can be represented in the following way:

$$F = \frac{F_0\{\alpha\sinh(\gamma gt) + \gamma\cosh(\gamma gt)\} - d\sinh(\gamma gt)}{F_0\sinh(\gamma gt)) - \alpha\sinh(\gamma gt) + \gamma\cosh(\gamma gt)},$$

where

$$\alpha = \frac{1}{2}(1 + d - a)\,,$$

and

$$\gamma = \frac{1}{2}\sqrt{(2\alpha)^2 - 4d}.$$

$A = \alpha + \gamma$ and $B = \alpha - \gamma$.

Let us look into the various value of F_0 in relation to A and B.

$B > F_0$.

In this case $F = 0$ (Eq. (3)) in a finite time interval:

$t = -\frac{1}{g(A-B)} \log \frac{A(B-F_0)}{B(A-F_0)}$.

Keep in mind $\frac{A(B-F_0)}{B(A-F_0)} < 1$, since $-A < -B$.

A question remains for $B > F_0$: does the denominator become 0 for a smaller value of t?

The denominator of Eq. (3) becomes zero for $t = -\frac{1}{g(A-B)} \log \frac{(B-F_0)}{(A-F_0)}$.

Now $\log \frac{A(B-F_0)}{B(A-F_0)} > \log \frac{(B-F_0)}{(A-F_0)}$. Hence the denominator becomes 0 for a time larger interval.

With $B > F_0$ we find with Eq. (5):

$a > 1 + d - F_0 - \frac{d}{F_0}$ and this catch leads to a vanishing population in a finite time interval.

So an additional condition for stable population is : $a < 1 + d - F_0 - \frac{d}{F_0}$.

This all looks rather subtle, since for $B = F_0$ we find with Eq.(3) : $F = B$. The population is independent of time, a constant.

Hence, a small variation of a makes the population fluctuate between a constant value independent of time and a vanishing value in a finite time interval.

There is something to add about the relation of F_0 and d for $B > F_0$. We have:

$a > 1 + d - F_0 - \frac{d}{F_0}$ and $(1 + d - 2\sqrt{d}) > a$.

This gives $1 + d - F_0 - \frac{d}{F_0} < a < 1 + d - 2\sqrt{d}$. (8)

Now we assume $F_0 + \frac{d}{F_0} < 1 + d$ in the above expression.(Otherwise we would have obtained nothing new: $0 < a < 1 + d - 2\sqrt{d}$).

After some algebra $F_0 + \frac{d}{F_0} < 1 + d$ gives $0 < F_0 - d < 1 - d$, where use has been made of $0 > d < 1$.

Then with Eq.(8) $F_0 + \frac{d}{F_0} > 2\sqrt{d}$: $(F_0 - \sqrt{d})^2 > 0$. This is always true except for $F_0 = \sqrt{d}$. For this particular value of $F_0 = \sqrt{d}$ Eq. (8) gives $1 + d - 2\sqrt{d} < a < 1 + d - 2\sqrt{d}$. Hence we conclude $F_0 > \sqrt{d}$. Whether this is a biological condition, I do not know.

We conclude this subsection with a stable population to be found for $B < F_0 < A$ and $(1 + d - 2\sqrt{d}) > a$.

With Eqs. (4) -(5) we obtain:

$2\sqrt{d} < F_0 + \frac{d}{F_0}$. This is the same as $(F_0 - \sqrt{d})^2 > 0$.

4.2 $\boldsymbol{a = 1 + d - 2\sqrt{d}}$.

Then, the special case $(1 + d - a)^2 - 4d = 0$. Obviously, we cannot use Eqs. (3) and (6).

The differential equation, with $a = 1 + d - 2\sqrt{d}$, changes into:

$\frac{1}{g}\frac{dF}{dt} = -[F - \sqrt{d}]^2$. (9)

We find F to be:

$$F = \frac{(F_0 - \sqrt{d})(1 + \sqrt{d}gt) + \sqrt{d}}{1 + \{F_0 - \sqrt{d}\}gt} . \tag{10}$$

This represent a hyperbole, starting at $F(t = 0) = F_0$. However, there is more to say. Since we can ask ourselves whether F can also be zero in a finite time interval. With Eq. (9) we find for $F = 0$:

$t = \frac{1}{g\sqrt{d}} \frac{F_0}{\sqrt{d} - F_0}$ and $\sqrt{d} > F_0$.

As explained by Strogatz, Eq. (9) represents an unstable case.

We could ask our self the question, does the solution of the range $(1 + d - 2\sqrt{d}) > a$ match the solution presented in Eq. (10)? So does a approaching $1 + d - 2\sqrt{d}$ in the solution for the smooth logistic curve gives the solution given in Eq. (10)? To find out we substitute in to Eq.(3)

$a + \varepsilon = 1 + d - 2\sqrt{d}$,

with $\varepsilon \to 0$. We plug this expression for a and ε in Eqs.(3)-(6). After collecting the relevant terms, we obtain:

$$F = \frac{(F_0 - \sqrt{d})(1 + \sqrt{d})gt + \sqrt{d}}{1 + (F_0 - \sqrt{d})gt}.$$

This expression equals Eq.(10).

We can look into this in a slightly different way:

a approaching $1 + d - 2\sqrt{d}$ gives with Eq. (4)-(5) : $A = \sqrt{d}$ and $B = \sqrt{d}$.

So in the 3-D space $F(a, t)$ we find a continuous function up to $a = 1 + d - 2\sqrt{d}$.

4.3 $(\mathbf{1} + \boldsymbol{d} - \mathbf{2}\sqrt{\boldsymbol{d}}) < \boldsymbol{a} < (\mathbf{1} + \boldsymbol{d} + \mathbf{2}\sqrt{\boldsymbol{d}})$

Next we look into the complex range Eq. (7) $(1 + d - 2\sqrt{d}) < a < (1 + d + 2\sqrt{d})$.

Well, a fisherperson(f/m) does not care much about complex numbers. Though fishing quota can be based on very, very complex regulations. These regulations can be very real indeed. Sorry, for the Trumpian hyperbole.

After carefully complex bookkeeping we have with Eq. (3):

$$F = \frac{F_0\{(\alpha \sin \omega t + \beta \cos(\omega t)\} - d \sin \omega t}{F_0 \sin \omega t - \alpha \sin \omega t + \beta \cos \omega t} , \tag{11}$$

where

$A = \alpha + i\beta$,

$B = \alpha - i\beta$,

$\alpha = \frac{1}{2}(1 + d - a)$,

$\beta = \frac{1}{2}\sqrt{4d - (2\alpha)^2}$,

$AB = \alpha^2 + \beta^2 = d$,

and $\omega = \beta g$.

F starts at F_0 for $t = 0$ and nosedives for $t > 0$ to $F = 0$ and $t = \frac{1}{\omega}\tan^{-1}(\frac{F_0\beta}{d - F_0\alpha})$.

Hence no complex catch.

Furthermore for $d - F_0\alpha \to 0$, $t \to \pi/2$.

From a biological point of view the range $(1+d-2\sqrt{d})<a<(1+d+2\sqrt{d})$ is an unacceptable catching range. Evolutionary this range makes no sense for prey- predator. Again, we could ask our self the question, does the solution of the range $(1+d-2\sqrt{d})<a$ match the solution presented in Eq. (10)? So does a approaching $1+d-2\sqrt{d}$ in Eq. (11) gives Eq. (10)? To find out we substitute in Eq.(11)

$a-\varepsilon=1+d-2\sqrt{d}$,

with $\varepsilon\to 0$.

After collecting the relevant terms, we obtain:

$F=\frac{(F_0-\sqrt{d})(1+\sqrt{d}gt)+\sqrt{d}}{1+(F_0-\sqrt{d})gt}$. Again, this expression equals Eq. (10).

So in the 3-D space $F(a,t)$ we find a continuous function up to $a>1+d-2\sqrt{d}$ and a approaching $1+d-2\sqrt{d}$.

We could ask ourselves whether at the other end of the interval $a<(1+d+2\sqrt{d})$ does the solution presented in Eq. (11) match the solution for $a=(1+d+2\sqrt{d})$? Before doing this analysis we first analyse $a=(1+d+2\sqrt{d})$.

4.4 $\boldsymbol{a=(1+d+2\sqrt{d})}$

So, we next analyse $a=(1+d+2\sqrt{d})$.

The differential equation is now:

$$\frac{1}{g}\frac{dF}{dt}=-[F+\sqrt{d}]^2. \tag{12}$$

Then we find F to be:

$$F=\frac{(F_0+\sqrt{d})(1-\sqrt{d}gt)-\sqrt{d}}{1+\{F_0+\sqrt{d}\}gt}. \tag{13}$$

Another hyperbolic expression.

Starting at $F(t=0)=F_0$.

In a finite time interval $F=0$. This time interval is

$t=\frac{1}{g\sqrt{d}}\frac{F_0}{F_0+\sqrt{d}}$.

Strogatz explains Eq. (12) again to represent an unstable situation similar to Eq. (9).

Now we can answer the question, does the solution of the range $\left(1+d+2\sqrt{d}\right)>a$ match the solution presented in Eq. (12)? So does a approaching $1+d+2\sqrt{d}$ in Eq. (10) gives Eq. (13)? To find out we substitute in Eq.(11)

$a+\varepsilon=1+d-2\sqrt{d}$,

with $\varepsilon\to 0$.

After collecting the relevant terms, we obtain:

$F=\frac{F_0-F_0\sqrt{d}(1+\sqrt{d})gt}{1+F_0(1+\sqrt{d})gt}$. Again, this expression equals Eq. (13).

In addition we expect the finite time interval giving $F=0$, $t=\frac{1}{g\sqrt{d}}\frac{F_0}{F_0+\sqrt{d}}$ to be smaller than the finite time interval resulting from Eq. (11), $t=\frac{1}{\omega}\tan^{-1}\frac{F_0\beta}{d-F_0\alpha}$. Hence we investigate :

$\frac{1}{g\sqrt{d}}\frac{F_0}{F_0+\sqrt{d}} < \frac{1}{\omega}\tan^{-1}\frac{F_0\beta}{d-F_0\alpha}$?

With $\omega = \beta g$, this inequality reads: $\frac{1}{\sqrt{d}}\frac{F_0\beta}{F_0+\sqrt{d}} < \tan^{-1}\frac{F_0\beta}{d-F_0\alpha}$. Since $\sqrt{d}\left(F_0 + \sqrt{d}\right) > d - F_0\alpha$, the inequality is true.

4.5 $\boldsymbol{a > (1 + d + 2\sqrt{d})}$

Now for the final range $a > (1 + d + 2\sqrt{d})$. No complex numbers. We have a smooth declining curve, starting at $F(t = 0) = F_0$ and for $\lim_{t\to\infty} F = A$. In this case A is a negative number for this range(See Eq. (4)). Hence this catching range is not allowable.
Furthermore: $A - B > 0$.
We can conclude a few things more from this range. Since F becomes zero, we can determine the time interval for this to happen.
Using Eq. (3) we find the time interval to be:

$$t = -\frac{1}{(A-B)g}\ln(-\frac{A}{CB}). \qquad (14)$$

So $0 < -\frac{A}{CB} < 1$. Since $A < 0$ and $B < 0$ C has to be smaller than zero.
With Eq. (6) $C = \frac{A-F_0}{F_0-B} < 0$ we have $\frac{A-B}{F_0-B} < 1$. Consequently $A - B < F_0 - B$. This leads to $A < F_0$. So we could also conclude for a given reproduction rate d and with Eq. (4):
$A = \frac{1}{2}[(1 + d - a) + \sqrt{(1 + d - a)^2 - 4d}] < F_0$. This will give us a constraint for the catching value a. This is nice. However, we already know the final catching range we dealt with to be not allowable.
Again, we could ask our self the question, does the solution of the range $\left(1 + d + 2\sqrt{d}\right) < a$ match the solution presented in Eq. (13)? So does a approaching $1 + d + 2\sqrt{d}$ in Eq. (3) gives Eq. (13)? To find out we substitute in Eqs. (3)-(6)

$$a - \varepsilon = 1 + d + 2\sqrt{d},$$

with $\varepsilon \to 0$.
After collecting the relevant terms, we obtain:
$F = \frac{F_0 - F_0\sqrt{d}(1+\sqrt{d})gt}{1+F_0(1+\sqrt{d})gt}$. Again, this expression equals Eq. (13).
To conclude the discussion of the interval for a we investigate whether t given in Eq. (14) is smaller than $\frac{1}{g\sqrt{d}}\frac{F_0}{F_0+\sqrt{d}}$ .The latter being the time interval for $a = (1 + d + 2\sqrt{d})$.

Remark:

Strogatz(chapter 3) discussed the model for insect outbreak and predation..
The model in dimensionless formulation*(Eq. 3, Chap.3)* reads:

$$\frac{dx}{d\tau} = rx(1 - \frac{x}{k}) - \frac{x^2}{1+x^2},$$

where τ represents time, r and k parameters. The exact solution of this differential equation can be found by factorization and separation of variables. In order to factorize we need the solution of quartic equation. (Abramowitz and Stegun).

5 Conclusion

We analysed the logistical equation Eq.(1).
For a growth factor g and reproduction factor d we investigated an modified Lotka-Volterra equation for the adjusted catch factor a.
The result $(1 + d - 2\sqrt{d}) > a$ is to give sustainable solutions for the catch factor a for a given reproduction factor d. For $a \geq 1 + d - 2\sqrt{d}$ the population becomes unstable.
a is not the real catch factor. a is the quotient of the catch factor c and the growth factor g.

6 Literature

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