On Special relativity and Classical Field Theory

The theoretical minimum Updated 2021-02-26, Edited Up to Exercise 6.2, page 241

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Remarks, Questions and Exercises.

Based on The Theoretical Minimum series by Susskind.

I adopt the Lecture System of Susskind.

Inhoudsopgave

Introduction

Susskind presented the postulates which Vol. III of the theoretical minimum series is based upon.
- The law of nature are the same in all frames of references.
- It is a law of nature that light moves with velocity c.
Consequently, a "*new theory of space and time*" came into existence.

Lecture 1 The Lorentz Transformation

1.1 Reference Frames

"*The special theory of relativity is about reference frames*".
Susskind starts with the assumption that the time coordinate is the same for every reference frame.

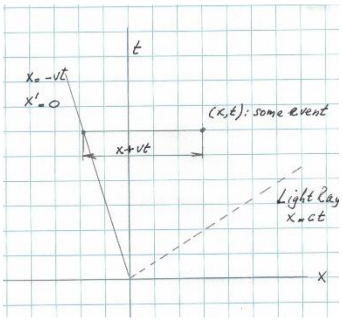
1.2 Inertial Reference Frames

"*The laws of Physics are the same in all inertial frames*".
page 6: *"…in general is that any two frames that are both inertial must move with uniform motion along a straight line".*

1.2.1 Newtonian (Pre-Special Relativity) Frames

Page 7: "*Newton's basic postulate would have been that there exists a universal time, the same in all reference frames*".
In this section Susskind analysed a rest frame and a moving frame with universal time.
A spacetime diagram, Fig 1.1, is presented. In this Figure, the velocity of the moving frame v, positive, and moving to the right.

Intermezzo: Spacetime diagram for a moving frame to the left



So, the motion of the moving frame is described by
$x = -vt$.
In the moving frame the motion is described by $x' = 0$.
End of intermezzo.

On page 10, Susskind introduced inversion of the system: to explain the event in different coordinates. So, inverting the relations described by Eqs. (1.1) and (1.2). The rest frame is moving. The event is at $(x', t') = (x', t)$.
Then, see Fig.(1.1),
$x = x' + vt'$, Eq.(1.4).
On page 11, Susskind explains the case where the light ray is moving to the left in the (x, t) frame
$x = -ct$.
Then, in the (x', t) frame, with Eq.(1.4), substitute $x = -ct$ into
$x = x' + vt \rightarrow -ct = x' + vt \rightarrow x' = -(c + v)t.$
The same result is obtained by substituting $c \rightarrow -c$
into
$x' = (c - v)t$.
Nota Bene: or t'.

On page 11, below $x' = -(c + v)t$, the contradiction is presented.
"…. *measuring the speed of light with great precision and found out that it's always the same, no matter how the inertial observer moves*".
So, Eqs. (1.1) and (1.2) need repair.

1.2.2 SR Frames
"The whole idea of simultaneity is frame dependent".

Synchronizing our clocks.
Susskind explained Einstein's thought experiment about synchronizing of clocks in one frame, the rest frame say. Then explained the clocks being synchronized, appears not to be synchronized in the moving frame when inspecting the clocks in the rest frame.
Units and Dimensions: A Quick Detour.
Two systems of units are used. The first is the system of "*conventional*" units: kg, m, sec system. The second system is based on the speed of light. The units are called *"relativistic"* units. In Fig. (1.2) a spacetime diagram is presented where use have been made of

relativistic units.

Setting Up Our Coordinates-Again.
The exact meaning of synchronous is presented. "*A line that represents the trajectory of an observer moving through spacetime is called a world line*".

Back to the Main Road.
Worldline and simultaneous events are analysed. Three worldlines for three moving frames are presented. All three move with the same velocity v at equidistant distances shown in Fig. (1.2). The result of simultaneous events are discussed and shown in Fig. (1.2).
Finding the x' Axis.
Keep in mind, in Fig. (1.2), $x' = 0$ is the time axis in the moving frame.
Near the bottom of page 20, Susskind represented the result for point a in Fig. (1.2). I denote this expression by Eq. (1.6a).
Eq. (1.7) is obtained by the summation of Eqs. (1.6) and (1.6a).
Finally the coordinates of point b, Fig (1.2), are obtained: Eqs. (1.8). From these equations it follows: $\frac{t_b}{x_b} = v$, Eq. (1.9): the x'-axis of the moving frame. In Fig. (1.3) the special relativity frames (SR Frames) are presented: the rest frame and the moving frame. In Fig. (1.4), the complete picture of the Sr Frames are presented without the construction of the moving frame. I recapitulate the findings of Susskind:
- *The pairs of events synchronous in one frame are not the same pairs that are synchronous in the other frame.*
- *In the moving frame, synchronicity corresponds to surfaces that are not horizontal, but are tilted with slope v*, being the velocity of the moving frame.

Space time.
In this paragraph, Susskind contemplated the findings of spacetime.

Lorentz Transformations.
This paragraph is about *the coordinate transformation relating the rest frame coordinates to the coordinates of the moving frame.*
Susskind starts with the Newtonian equation and multiplies it on the right-hand side with a function of the velocity of the moving frame. Here, I would say again, symmetry between right and left is used leading to Eq. (1.13): there is no preferred direction in space.
The first step in the transformation equations is given: Eq. (1.15). to figure out the Lorentz transformations, Susskind implied Einstein's principle that the speed of light is the same in the moving frame as well in the rest frame.
So, use is made of $x = t$, and $x' = t'$. Then, Eq. (1.15) can be simplified: Eqs. (1.16).
Next, use has been made of the symmetry between the rest frame and the moving frame. This results into Eq.1.17). Susskind plugged the expressions for x' and t' from Eqs. (1.16) into the first equation of (1.17). This finally gives us the Lorentz Transformations: Eqs.(1.19) and (1.20).
To illustrate another symmetry, I plugged the expressions for x' and t' from Eqs.(1.16) into the second equation of (1.17):

$t = (t' + vx')f(v^2) \rightarrow t = \{(t - vx)f(v^2) + v(v - xt)f(v^2)\}f(v^2) =$
$= (t - vx)f(v^2)f(v^2) - v(x - vt)f(v^2)f(v^2) =$
$= tf(v^2)f(v^2) - vxf(v^2)f(v^2) + vxf(v^2)f(v^2) - v^2tf(v^2)f(v^2) =$
$= tf(v^2)f(v^2) - v^2tf(v^2)f(v^2) = t(1 - v^2)f(v^2)f(v^2).$
$t = t(1 - v^2)f(v^2)f(v^2).$
Et voila:
$f(v^2) = \frac{1}{\sqrt{1-v^2}}.$
Giving the Lorentz Transformations: Eqs.(1.19) and (1.20)

1.2.3 Historical aside

In this section Susskind presented some historical background of the Lorentz Transformations.

1.2.4 Back to the Equations.

In this section, Susskind used symmetry(again) to the derive the twins of Eqs. (1.19) and (1.20). Meaning, expressing the coordinates of the rest frame, (x, t), into the coordinates of the moving frame→Eqs. (1.21) and (1.22).

Switching to conventional units.

Conventional units: $c \neq 1$.

To switch to conventional units is to take care the equations to be dimensionally consistent. In Eqs. (1.23) and (1.24), the Lorentz transformations in Eqs. (1.19) and (1.20), are presented in conventional units.

The Other Two Axes.

In Eqs. (1.25) -(1.28), the complete set of Lorentz transformations for moving with velocity v in the position direction, are presented.

1.2.5 Nothing Moves Faster than Light.

The Lorentz transformations clearly indicate the impossibility to move faster than light.

1.3 General Lorentz Transformation.

On page 37, presented the three steps to change general Lorentz transformations into simple Lorentz transformations.

First, rotate space to align primed axes with unprimed axes. Then, apply (simple) Lorentz transformations along the new x-axis. Finally, rotate space to restore the original orientation.

1.4 Length Contraction and Time Dilation.

Susskind started this section with the important advice: to analyse special relativity- draw a spacetime diagram.

Length Contraction.

Measure the length of a meterstick in the rest frame, while being in the moving frame. The endpoints of the meterstick should be measured at the same time, simultaneously, in the moving frame at $t' = 0$. This case is illustrated in the spacetime diagram of Fig. (1.5).

In the rest frame the length of the meterstick is 1m at any time. So, Susskind draws the vertical at Q in the rest frame. The Lorentz transformation given in Eq. (1.20) for $t' = 0$:

$t' = \frac{t-vx}{\sqrt{1-v^2}} \rightarrow \frac{t-vx}{\sqrt{1-v^2}} = 0 \rightarrow t - vx = 0 \rightarrow t = vx.$
Plug $t = vx$ into Eq. (1.19): →with $x = 1 \rightarrow x' = \sqrt{1-v^2}$.
This represents a shorter meterstick in the moving frame.

Remark: the contracted length is $\sqrt{1-v^2}$. The length contraction is: $1 - \sqrt{1-v^2}$. Subtle or futile?

Exercise 1.1 Length contraction in the rest frame.

Show that the x-coordinate of point Q in the rest frame, Fig. (1.6), is $\sqrt{1-v^2}$.
Fig. (1.6): Susskind presented here the spacetime diagram. I will use it.
Simultaneity in the rest frame is at $t = 0$.
In the moving frame the end point of the meterstick is at point P and it stays there at any time. So, draw the tilted line at $x' = 1$, parallel to the line $x' = 0$. At $t = 0$, the $x' = 1$ crosses at point Q.
Eq.(1.22):
$t = \frac{t'+vx'}{\sqrt{1-v^2}} \rightarrow \frac{t'+vx'}{\sqrt{1-v^2}} = 0 \rightarrow t' + vx' = 0 \rightarrow t' = -vx'.$
Plug $t' = -vx'$ into Eq.(1.21): → $x = \frac{x'-v^2x'}{\sqrt{1-v^2}}$.
We know $x' = 1$m→ $x = \frac{1-v^2}{\sqrt{1-v^2}} = \sqrt{1-v^2}$.
So, the position of Q is at $x = \sqrt{1-v^2}$.

Time Dilation.
Now, Susskind used a clock to find out about time dilation. The clock indicates the time $t' = 1$, in the moving frame. What about the time in the rest frame?
In the rest frame, simultaneity means a line parallel with the x-axis through $t' = 1$, giving t on the $x = 0$, axis, Fig. (1.7).
The result for t is found with the Lorentz transformation given in Eq. (1.22), with $t' = 1$ and $x' = 0, \rightarrow t = \frac{1}{\sqrt{1-v^2}}$.

Remark: the dilated time is $\frac{1}{\sqrt{1-v^2}}$. The time dilation is: $\frac{1}{\sqrt{1-v^2}} - 1$. Subtle or futile?
Furthermore, the clock is in the moving frame. I expected to use the line of simultaneity in the moving frame. Meaning to use the line parallel to the axis $t' = 0$, through $t' = 1$. Cutting the line $x = 0$, below t in Fig. (1.7).

Now, let's find out about symmetry with a clock in the rest frame at $t = 0$. Following the procedure as given in Fig. (1.7), now we use the line of simultaneity in the moving frame. Meaning to use the line parallel to the axis $t' = 0$, through $t = 1$.
Then with Eq. (1.20), with $t = 1$, and $x = 0$:
$t' = \frac{t-vx}{\sqrt{1-v^2}} \rightarrow \frac{1}{\sqrt{1-v^2}}$.
So changing from the moving frame into the rest frame resulted into the same dilated time, showing the symmetry.

The Twin Paradox

Two observers, one in the rest frame and the other in the moving frame. In Fig. (1.8), the case is illustrated. After a certain time, the observer in the moving frame returns to the rest frame starting position at $x = 0$. Susskind explains in this case the symmetry got lost. We are no longer considering inertial frames.

Exercise 1.2: About the Twin Paradox.

In Fig. (1.8), the traveling twin not only reverse direction but switches to a different reference frame when the reversal happens.

a) Use Lorentz transformation to show that before the reversal happens, the relationship between the twins is symmetrical. Each twin sees the other as aging more slowly than himself.

Susskind presented the result for the clock in the moving frame: $t = \frac{1}{\sqrt{1-v^2}}$. Above, I obtained the result for the rest frame: $t' = \frac{1}{\sqrt{1-v^2}}$.

Consequently, before the reverse the relationship between the twins is symmetrical.

b) Use spacetime diagrams to show how the traveller’s abrupt switch from one frame to another changes his definition of simultaneity.

In the Figure on the next page, I used the basic case as presented by Susskind in Fig. (1.8). I will analyse the twin paradox from the perspective of the moving twin (MT). The twin in the rest frame is denoted by RT.

Fig. (1.8), $t_r = \frac{1}{\sqrt{1-v^2}}$.

The RT worldline is $x = 0$. The worldline of MT, just before $t' = 1$, is $x = vt$.

Just before the turnaround at $t' = 1$, the MT calculates the “age” of the RT., indicated by the intersection of the line of simultaneity, l_2, and $x = 0$:

$$t_2 = t_r - \Delta t = \frac{1}{\sqrt{1-v^2}} - \Delta t. \qquad \text{(C.1.4.1)}$$

Note, I will pay attention to Δt later.

Just after the reversal of the moving back to the $x = 0$-axis, MT calculates the “age” of RT.

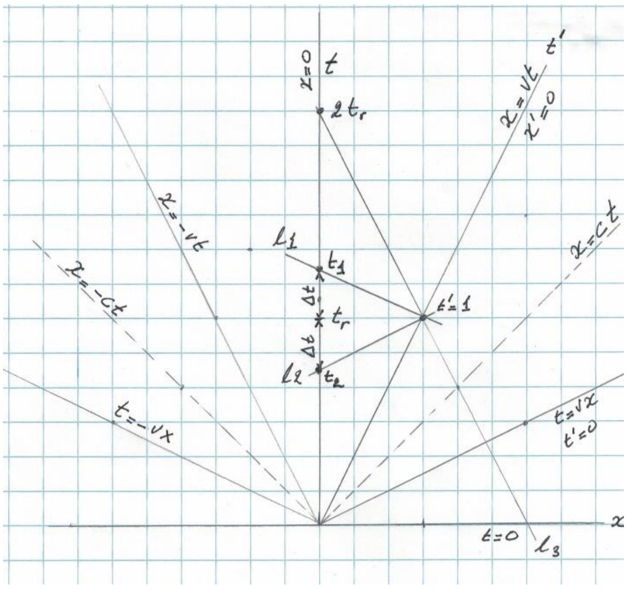


Figure 1 The Twin paradox, Fig1.8 page 44

Now the line of simultaneity is l_1, where I used symmetry. This line intersects the $x = 0$-axis at:

$t_1 = t_r + \Delta t = \frac{1}{\sqrt{1-v^2}} + \Delta t.$ (C.1.4.2)

A jump between two lines of simultaneity for MT and a jump in two worldlines for MT. This leads to a jump in age for RT, in the rest frame of $2 \cdot \Delta t$.

The new worldline for MT is l_3.

Can I find out about $t_1 - t_2 = 2 \cdot \Delta t$? I give it a try. However, I must keep in mind the remarks of Susskind below Exercise 1.2 with respect to numbers and the visual picture:

$t_r = \frac{1}{\sqrt{1-v^2}}$ is larger, in Fig. (1.8), than $t' = 1$. So, am I allowed to use $x = -vt$ to be perpendicular to $t = vx$? I think it can be done.

First, returning to earth, the age of MT is 2 , and the age of RT is $\frac{2}{\sqrt{1-v^2}}$.

Then the aging of RT is:

$\frac{2}{\sqrt{1-v^2}} - 2 = 2\frac{1-\sqrt{1-v^2}}{\sqrt{1-v^2}}$,

and

$\Delta t = \frac{1-\sqrt{1-v^2}}{\sqrt{1-v^2}}$. (C.1.4.3)

Is this correct?

Next, the equation for worldline l_3. This worldline is parallel to $x = -vt$. The worldline l_3 is shifted over some distance represented by a constant c , giving

$x = -vt + c$. (C.1.4.4)

We know at $x = 0$, $t = 2t_r = \frac{2}{\sqrt{1-v^2}}$.

Substitute $t = 2t_r = \frac{2}{\sqrt{1-v^2}}$, into (C.1.4.4):

$x = -vt + \frac{2v}{\sqrt{1-v^2}}$. (C.1.4.5)

Now, the caveat of Susskind. I call it geometrical fallacy.

Let's use geometry:

The length L of the worldline with the negative slope is:

$L^2 = (2t_r)^2 + (\frac{v}{\sqrt{1-v^2}})^2 \Leftrightarrow \frac{5}{1-v^2} \rightarrow L = \sqrt{\frac{5}{1-v^2}}$. (C.1.4.6)

Then, from the geometrical analysis I find an expression for Δt, with (C.1.4.6) and t_r:

$\frac{t_r+\Delta t}{L/2} = \frac{L}{2t_r} \rightarrow \Delta t = \frac{1}{4\sqrt{1-v^2}}$.

Compare this result with (C.1.4.3) and the fallacy is clear.

The caveat with use of geometry in spacetime diagrams, I encountered in this Exercise 1.2. Therefore, I doubt the parallelism between l_2 and, for example $t = vx$. Well, at least in this 2-domensional diagram. So, I suppose it is about curved surfaces of simultaneity.

The Stretch Limo and the Bug

it is about length contraction. Basically, about simultaneity. The question to be dealt with is: "*Simultaneous according to whom?*".

1.5 Minkowski's World

In this section, the concept of invariance is explained.
Susskind defined the invariant quantity: "…*Some aspect of spacetime that has the same value in every reference frame*."
Susskind illustrated invariance in Fig.1.10.
Then, the "geometrical problem", observed in Exercise 1.2 and illustrated in Fig.1.8, is discussed. In the diagram $\frac{1}{\sqrt{1-v^2}} < 1$. That is impossible. Consequently, Euclidean geometry does not work.
Susskind asked: *"Is there an invariant in Minkowski's space?".* On page 51-53, Susskind investigated a possible invariant quantity.
In Eq.(1.29):

$$t'^2 - x'^2 = \frac{t^2+v^2x^2}{1-v^2} - \frac{x^2+v^2t^2}{1-v^2} = \frac{t^2-v^2t^2}{1-v^2} - \frac{x^2-v^2x^2}{1-v^2} = t^2\frac{1-v^2}{1-v^2} - x^2\frac{1-v^2}{1-v^2} = t^2 - x^2 \,,$$

Eq.(1.30).
In Eq.(1.31), Susskind presented the generalized invariant: the proper time τ.

1.5.1 Minkowski and the Light Cone

Susskind started this section with an investigation into the history of proper time. "*It is to Minkowski that we owe the concept of time as the fourth dimension of a four-dimensional spacetime".*
The light cone, described by Eq. (1.32) consist of a future light cone and a past light cone, Fig.1.11.

1.5.2 The Physical Meaning of Proper Time

Th e physical meaning is formulated on page 57: "*The invariant proper time along a worldline represents the ticking clock moving along that worldline*".

1.5.4 Timelike, Spacelike, and Lightlike Separations

The separations may be based on τ^2 and s^2.
Susskind analysed these separations with a light signal. The Light cone is used and the light flash is at the origin, Fig.1.11.

Timelike Separation
Definition: $t^2 > x^2 + y^2 + z^2$.
As in Fig.1.11: $t^2 > x^2 + y^2$.
This is about an event within the cone.
The property of being timelike is invariant.

Spacelike Separation
Definition: $t^2 < x^2 + y^2 + z^2$.
This is about an event outside the cone.
The spacelike property is invariant.

Lightlike Separation
Definition: $t^2 = x^2 + y^2 + z^2$.
An event on the cone
The lightlike property is invariant.

1.6 Historical Perspective

1.6.1 Einstein

Einstein analysed Maxwell's equations. The important finding of Einstein was to find out the symmetry structure of Maxwell's equations is the Lorentz Transformation.

1.6.2 Lorentz

Lorentz interpreted the transformation equations as effects on moving objects through the ether.

Lecture 2 On Velocities and 4-vectors

In the introductory remarks, Susskind paid attention to the possibility of moving faster than light by adding velocities.

2.1 Adding velocities.

Fig.2.1 illustrates the case Susskind analyses. It is about three observers: A in the rest frame, L in the moving frame, and M in a moving frame moving in the frame of L.
How to find out about the relative velocities? How to use the Lorentz transformations?

2.1.1 The third observer M

The velocity of M relative to L is u. Lorentz transformation are used to connect L-and M's coordinates. Knowing these transformations, the relations between A- and M's coordinates can be found.
In Eq. (2.8) Susskind presented the velocity of a worldline. It is the velocity of M in A's rest frame. This velocity has been obtained by the equation of M's worldline: $x'' = 0$.

Reminder: in Lecture 1 we learned the worldline of the moving frame to be $x = vt$ for $x' = 0$. This is about two observers, one in the rest frame and one in the moving frame.

Exercice 2.1 Lorentz Transformations relating A-and M's frames.

I denoted this Exercise 2.1.
Susskind already derived the relation for x'', Eq.(2.7) and the expression for a worldline moving with w , Eq.(2.9), in A's frame-the rest frame.
Now

$$t'' = \frac{t' - ux'}{\sqrt{1-u^2}} .$$

Plug into this expression:

$$t' = \frac{t - vx}{\sqrt{1-v^2}} ,$$

and

$$x' = \frac{x - vt}{\sqrt{1-v^2}} .$$

Then

$$t'' = \frac{t - vx - u(x - vt)}{\sqrt{1-u^2}\sqrt{1-v^2}} = \frac{t(1+uv) - x(u+v)}{\sqrt{1-u^2}\sqrt{1-v^2}} . \qquad \text{(C.2.1.1.1)}$$

Since Susskind already derived the expression for
$w = \frac{u+v}{1+uv}$, with (C.2.1.1.1):

$$t'' = (1 + uv)\frac{t - wx}{\sqrt{1-u^2}\sqrt{1-v^2}} . \qquad \text{(C.2.1.1.2)}$$

We know $w = \frac{u+v}{1+uv}$, so the question to be answered is:
$\frac{(1+uv)}{\sqrt{1-u^2}\sqrt{1-v^2}} = \frac{1}{\sqrt{1-w^2}}$?
Let us evaluate:
$$\frac{(1+uv)}{\sqrt{1-u^2}\sqrt{1-v^2}} = \frac{1+uv}{\sqrt{1-v^2-u^2+v^2u^2}} = \frac{1+uv}{\sqrt{1+2uv+v^2u^2-v^2-2vu-u^2}} = \frac{1+uv}{\sqrt{(1+uv)^2-(u+v)^2}} =$$
$$\frac{1}{\sqrt{1-(\frac{u+v}{1+uv})^2}} = \frac{1}{\sqrt{1-w^2}}.$$
Hence, with (C.2.1.1.2):
$$t'' = \frac{t-wx}{\sqrt{1-w^2}}.$$
Similarly, $x'' = \frac{x-wt}{\sqrt{1-w^2}}$, is obtained.

To show the effect of the Lorentz double transformations, Susskind presented some numerical examples on pages 68 and 69.

2.2 Light Cones and 4-Vectors

A reminder:
- Proper time $\rightarrow \tau^2 = t^2 - (x^2 + y^2 + z^2)$ is an invariant,
- Spacetime interval relative to the origin $\rightarrow s^2 = -t^2 + (x^2 + y^2 + z^2)$ is an invariant, under Lorentz transformations.

2.2.1 How Light Rays Move

Susskind: *"…. A light ray moves in such a way that the proper time along its trajectory is zero"* $\rightarrow t = x$.

2.2.2 Introduction to 4-Vectors.

Basically, it is about an interval between two points in space.
New Notation.
The notation for a- 4 vector: X^μ . Furthermore, X^i is a vector representing the special components of the 4-vector.
On page 73, proper time and spacetime are presented in the new notation.
On page 74, the Lorentz transformations are presented in the new notation.
4-Velocity.
In this section 4-velocity is derived. Fig.2.3 illustrates a trajectory of a particle. The important difference with the usual derivation of velocity $\vec{v} = \frac{\Delta\vec{x}}{\Delta t}$, the interval to be divided by $\Delta\tau$. Susskind: "*The reason is that $\Delta\tau$ is invariant*".

Lecture 3 Relativistic Laws of Motion

Items dealt with are energy, momentum, canonical momenta, Hamiltonians and Lagrangians, the principal of least action included.

3.1 More About Intervals

Shorthand for the spacetime interval is presented by Eq. (3.1).

3.1.1 Spacelike Intervals

In this section Susskind illustrates the possibility, through Lorentz transformations, an event in one frame change from happening later to earlier in another frame.
Fig.3.1: in the t-x frame event a happens before event b, in the t'-x' frame b happens before a .

3.1.2 Timelike Intervals

In Fig.3.2 an example of a timelike interval is shown⇒the velocity can never reach the speed of light.

3.2 A Slower Look at 4-Velocity

Susskind asked the question on page 85: *What is the connection between 4-velocity and ordinary velocity?*
In this way, the time component of the 4-velocity of a moving observer is derived. Susskind presented the Newtonian limit. Next the spatial components are considered. In the Newtonian limit *the space components of the relativistic velocity are practically the as the components of the ordinary 3-velocity.*
On page 88, the 4-velocity summary is presented.

Exercise 3.1 Use of the definition of $\Delta\tau^2$

$$\Delta\tau^2 = (\Delta t)^2 - (\Delta\vec{x})^2 \Rightarrow \left(\frac{\Delta\tau}{\Delta t}\right)^2 = 1 - \left(\frac{\Delta\vec{x}}{\Delta t}\right)^2 \rightarrow \frac{d\tau}{dt} = \sqrt{1 - v^2}.$$

Eqs.(3.8)-(3.10):

$$(U^0)^2 = \frac{1}{1-v^2},$$

$$(U^i)^2 = \frac{(V^i)^2}{1-v^2}.$$

Eq.(3.7):

$$(U^0)^2 - \sum_i \left(U^i\right)^2 = 1 \Leftrightarrow \frac{1}{1-v^2} - \sum_i \frac{\left(V^i\right)^2}{1-v^2} \Leftrightarrow \frac{1}{1-v^2} - \frac{v^2}{1-v^2} \Leftrightarrow \frac{1-v^2}{1-v^2} = 1.$$

3.3 Mathematical Interlude: An Approximation Tool

Approximation methods are explained. A helpful tool to find analytical solutions of relevant equations. These solutions can be used as a first estimate to use numerical methods.
Eqs. (3.18) and (3.19) presents the first order approximations of:

$$(1 - v^2)^{\pm\frac{1}{2}}.$$

3.4 Particle Mechanics

It is about the position or velocity of the centre of mass of a particle.

3.4.1 Principle of Least Action.

Note: See Volume I, The Theoretical Minimum.

The Principle is the most central idea in all of physics.
All the laws of Physics are based on the action Principle. The Principle relates to energy conservation and momentum conservation.
Susskind briefly reviews the Principle.

3.4.2 A Quick Review of Nonrelativistic Action.

In Eq. (3.20), the action integral of the Lagrangian is given. Eq. (3.21) presents the simplest Lagrangian: the kinetic energy for a nonrelativistic particle.

3.4.3 Relativistic Action

Conditio sine qua non: the action should be an invariant.
The invariant is proper time separating the two positions between the particle moves.
At the top of page 97< Susskind explained once more the need for proper time.
In Eq. (3.23), the relativistic action integral has been derived.
Eq. (3.27) presents the action integral in conventional units.

3.4.4 Nonrelativistic Limit.

For small velocities, $\frac{v}{c} \ll 1$, the classical Lagrangian must be obtained. The approximations presented in Eqs. (3.16) and (3.17) are applied.
Plug the approximation given by Eq. (3.16) into Eq. (3.26): the kinetic energy from Newtonian mechanics is recovered.

3.4.5 Relativistic Momentum.

Susskind described the method Einstein used to find out about momentum.
In this section Susskind used the Lagrangian presented in Eq. (3.27), in conventional units.
Eq. (3.28), the expression for momentum:

$$P^i = \frac{\partial \mathcal{L}}{\partial \dot{X}^i}.$$

The x-component of $\frac{\partial \mathcal{L}}{\partial \dot{X}^i}$, with Eq.(3.27):

$$P^x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial}{\partial \dot{x}}\left[-mc^2\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}\right] = m\frac{\dot{x}}{\sqrt{1-\frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{c^2}}} \Rightarrow \text{in general Eq.(3.30).}$$

Then, Susskind compared the result presented in Eq.(3.30) with the nonrelativistic momentum.
In Eq.(3.9), the relativistic velocity derived from the definition is given. Compare this with the result in Eq. (3.30) ⇒the relativistic momentum is the mass times the relativistic velocity: Eq. (3.31).
to conclude this section: $P^i \to \infty$, for $v \to c$.
Eq. (3.32): $P = \int_{t_0}^{t} F dt' \ \therefore P \to \infty \Rightarrow t \to \infty$.

3.5 Relativistic Energy

Energy, another conserved quantity. The Hamiltonian comes into play in Eq. (3.35).
Susskind, page 106,: "**The three components of spatial momentum P^i together with the energy P^0 form a 4-vector".**

3.5.1 Slow Particles

In this section Susskind relates the relativistic Hamiltonian/energy with the nonrelativistic energy. The energy for $\frac{v}{c} \ll 1$, Eq. (3.39) is derived. At the end of this section attention is paid to the remark at the end of section 3.4.4: $E = mc^2$.

Terminology: Mass and Rest Mass
Briefly: Rest Mass≡Mass.

3.5.2 Massless Particles.
An example of a massless particle is a Photon. Massless particles move with velocity c.
The relativistic energy of a massless particle, Eq. (3.26)
$P^0 = mU^0$ ⇒the energy E.
So,
$m^2(U^0)^2 = E^2$.
Then, multiply Eq.(3.10) with m^2:
$m^2(U^0)^2 - m^2(\vec{U})^2 = m^2$, or,
$E^2 - P^2 = m^2$, (Eq,(3.42).
Finally, Eq.(3.45) represents the energy of a massless particle.

3.5.3 An Example: Positronium Decay
The decay of Positronium, an electron and a positron, into two photons is described by using energy and momentum conservation.

Lecture 4 Classical Field Theory
Vector fields and scalar fields are combined.

4.1 Fields and Spacetime
The way fields are represented is explained. The representation is given in the form:
$\phi(t, X^i)$, page 117.

4.2 Fields and Action
The study of fields will be based on the action principle.

4.2.1 Nonrelativistic Particles Redux
Susskind first considers a field without (or zero) space dimensions: $\phi(t)$.
The action integral is derived: Eq. (4.2).
Newton's equation of motion for a particle is obtained: Eq. (4.3).

4.3 Principles of Field Theory
In this Lecture, space dimensions are plugged into $\phi(t)$.

4.3.1 The Action Principle
Susskind:
"*The general problem of field theory can be stated in the following way: Given the values of ϕ everywhere on the boundary of the spacetime box, determine the field everywhere inside the box*".
Then, find $\phi(t, x, y, z)$, inside the box, that gives the least action:
$Action = \int \mathcal{L} d^4x$.

4.3.2 Stationary Action for ϕ
$\mathcal{L}$ depends on ϕ and the partial derivatives of ϕ.
Eq. (4.5) is the Euler-Lagrange equation for a single field.

Eq. (4.6) are the Euler-Lagrange equations for two fields.

Remark: at the top of page 126, Susskind mentioned
$\sum_\mu \frac{\partial}{\partial X^\mu}\frac{\partial \mathcal{L}}{\partial(\frac{\partial\phi}{\partial X^\mu})}$, consists of 4 terms ⇒ the summation over μ . I may conclude $\frac{\partial}{\partial X^2}\frac{\partial \mathcal{L}}{\partial(\frac{\partial\phi}{\partial X^1})} = 0?$
This is one of the twelve terms, all to be zero? I suppose so.

At the end of this section, Susskind mentioned not to have been specific whether ϕ is a scalar or a vector:
- a scalar has one component ⇒ one Euler-Lagrange equation,
- a vector has more components ⇒ every component a Euler-Lagrange equation.

4.3.3 More About Euler-Lagrange Equations

After some discussion, the kinetic and potential energy are presented in the Lagrangian: Eq. (4.7).
On pages 128-130, Susskind analysed Eq. (4.5) step by step.
Step 1: the derivative of $\mathcal{L}$ with respect to $\frac{\partial\phi}{\partial t}$, with Eq. (4.7) ⇒ $\frac{\partial\phi}{\partial t}$.
Keep in mind: $t \Leftrightarrow X^0$.
Step 3: $-\frac{\partial\mathcal{L}}{\partial\phi} = \frac{\partial V}{\partial\phi}$, page 120.
The equation of motion in conventional units is presented in Eq.(4.9).

4.3.4 Waves and Wave Equations

Susskind: "*This connection between field theory and wave motion is one of the most important in physics".*
A family of solutions of the simplest wave equation:
$\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial x^2}$, Eq. (4.10).
Susskind showed the class of function $F(x+ct)$. This function "*rigidly moves to the left with velocity c*".
Now the right-moving waves: $F(x-ct)$.
$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial(x-ct)}\frac{\partial(x-ct)}{\partial t} = -c\frac{\partial F}{\partial(x-ct)}$.
Then,
$\frac{\partial^2 F}{\partial t^2} = -c\frac{\partial^2 F}{\partial(x-ct)^2}\frac{\partial(x-ct)}{\partial t} = c^2\frac{\partial^2 F}{\partial(x-ct)^2} \Rightarrow \frac{\partial^2 F}{\partial(x-ct)^2} = \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}$.
Similarly,
$\frac{\partial^2 F}{\partial x^2} = \frac{\partial^2 F}{\partial(x-ct)^2}\frac{\partial(x-ct)}{\partial c} = \frac{\partial^2 F}{\partial(x-ct)^2}$.
Hence,
$\frac{1}{c^2}\frac{\partial^2 F}{\partial t^2} = \frac{\partial^2 F}{\partial x^2}$.

4.4 Relativistic Fields

Defining the action, proper time is chosen as the invariant.
So, "the *equations of motion using stationary action principle have exactly the same form in every reference frame*".
Susskind explained the need for a clear concept of the transformation properties of fields.

4.4.1. Field Transformation Properties

A scalar field is defined in Eq. (4.12).
An example of a vector field is the wind velocity.
Then, Susskind dealt with the 4-vector fields, Eq. (4.14).
The spacetime gradients of the scalar field ϕ are presented, Eq. (4.14).
Note: Page 136, just above section 4.4.2, ϕ_μ should read $\partial_\mu \phi$. A typo I suppose.

4.4.2 Mathematical Interlude: Covariant Components

Eq. (4.15) is centre stage: the relation between two infinitesimal intervals, primed and unprimed.
Einstein's summation convention is introduced, Eq. (4.16).
The partial derivatives, using the Lorentz transformations Eq. (4.17), for an infinitesimal interval are presented Eq. (4.18).
With help of the equations, Susskind showed the generalization for any 4-vector: Eqs. (4.19) and (4.20). Now, the transformation of $\partial_\mu \phi$ can be derived. Again, the chain rule is used, bottom page 139.
Eq. (4.14): on the left-hand side it is the primed derivative of $\phi \Rightarrow \partial'_\nu$. It is still about a scalar field $\phi \Rightarrow \phi' = \phi$.
On page 140, an important difference between derivatives and the generalization is explained:
- contravariant components A^ν, with superscripts,
- covariant components, A_μ , with subscripts,

and

the difference between coefficients.
On page 141 and 142, Susskind presented the difference between the coefficients by using the Lorentz transformations: *"All we have to do is to interchange primed and un primed coordinates and at the same time reverse the sign of the velocity v."*
The transformation rules for the components of the covariant 4-vectors are presented in Eq.(4.23).
I table 4.1, page 144, Field Transformations are summarized.
As suggested by Susskind, I present here as an exercise the demonstration of the invariant of a general 4-vector.

Exercise 4.1 The scalar, the invariant, for a general contravariant 4-vector

Demonstrate $(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2$ to be a scalar: an invariant.
To demonstrate this I will use the transformation factors of Table 4.1. Similar to the approach of pages 50 and 51.

$$(A'^0)^2 - \left(A'^1\right)^2 - \left(A'^2\right)^2 - \left(A'^3\right)^2 = \left(\frac{A^0 - vA^1}{\sqrt{1-v^2}}\right)^2 - \left(\frac{A^1 - vA^0}{\sqrt{1-v^2}}\right)^2 - (A^2)^2 - (A^3)^2 =$$

$$= \frac{\left(A^0\right)^2 - 2A^0A^1 + v^2\left(A^1\right)^2}{1-v^2} - \frac{\left(A^1\right)^2 - 2A^1A^0 + v^2\left(A^0\right)^2}{1-v^2} - (A^2)^2 - (A^3)^2 =$$

$$\frac{\left(A^0\right)^2 - v^2\left(A^0\right)^2}{1-v^2} - \frac{\left(A^1\right)^2 - v^2\left(A^1\right)^2}{1-v^2} - \frac{2A^0A^1 - 2A^1A^0}{1-v^2} - (A^2)^2 - (A^3)^2 =$$

$$(A^0)^2 - (A^1)^2 - (A^2)^2 - (A^3)^2.$$

End of demonstration.

4.4.3 Building a Relativistic Lagrangian

The Lagrangian must be the same in every coordinate frame ⇒ a scalar.
Then, Susskind presented some candidate building blocks for the Lagrangian., pages 145 and 146.

4.4.4 Using Our Lagrangian

In Eq. (4.24), a Lagrangian is presented and subsequently analysed. This Lagrangian is invariant under a general Lorentz transformation.

4.4.5 Classical Field Summary

In this section, Susskind presented the recipe for building the Lagrangian, subsequently apply the Euler-Lagrange equations. Then study the resulting wave equation.

4.5 Fields and Particles-A Taste

Susskind reflected on the relation between particles and fields.
The *Action* integral is given in Eq. (4.27).
In Eq. (4.28), a constructed *Action* integral is presented.
The Lagrangian is, Eq. (4.29):

$$\mathcal{L} = -[m + \phi(t,x)]\sqrt{1 - v^2} \,.$$

4.5.1 The Mystery Field

With the example chosen in Eq. (4.28) or Eq. (4.29):

$$\mathcal{L} = -[m + \phi(t,x)]\sqrt{1 - v^2} \,,$$

- A scalar field $\phi(t,x)$ shifts the mass of a particle,
- The Higgs field, the mystery field.

4.5.2 Some Nuts and Bolts

The Euler-Lagrange equation is derived for the Lagrangian presented in Eq. (4.29) and adjusted for the one-dimensional case: $v \Rightarrow \dot{x}$:

$$\mathcal{L} = -[m + \phi(t,x)]\sqrt{1 - \dot{x}^2} \,.$$

So,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = -[m + \phi(t,x)]\frac{\partial}{\partial \dot{x}}\sqrt{1 - \dot{x}^2} \Rightarrow \text{Eq.(4.30)}.$$

The other expression of the Euler-Lagrange equation is obtained from:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{d}{dt}\frac{[m+\phi(t,x)]\dot{x}}{\sqrt{1-\dot{x}^2}} \,.$$

The resulting Euler-Lagrange equation is presented at the top of Page 154.
In conventional units:

$$\frac{d}{dt}\frac{[m+\phi(t,x)]\dot{x}/c}{\sqrt{1-(\dot{x}/c)^2}} = -c\frac{\partial \phi}{\partial x}\sqrt{1-(\dot{x}/c)^2} \,.$$

For $\frac{\dot{x}}{c} \ll 1$:

$$\sqrt{1-(\dot{x}/c)^2} \approx 1 - \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2 \,.$$

$$\frac{d}{dt}[m + \phi(t,x)]\dot{x}/c[1 + \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2] = -\frac{\partial \phi}{\partial x}[1 - \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2] \,,$$

up to $\left(\frac{\dot{x}}{c}\right)^3$, this expression becomes:

$\frac{d}{dt}[m + \phi(t,x)]\dot{x}/c = -\frac{\partial\phi}{\partial x}[1 - \frac{1}{2}\left(\frac{\dot{x}}{c}\right)^2]$.
m and $\phi(t,x)$ have the dimension of kg.

Lecture 5 Particles and Fields

The central question: *"If a field affects a particle, for example by creating forces on it, must a particle affect the field?"*
There can be interaction.

5.1 Field Affects Particle (Review)

There exists a field $\phi(t,x)$.
As mentioned in Lecture 4, the particle is coupled to the field as presented by Eq. (4.28) and reproduced at the top of page 159.
Since the nonrelativistic limit is considered, the action integral is given in conservative units. The Lagrangian is given in units of energy. The result of that is given on page 160.

Remark: on page 154, I considered m and $\phi(t,x)$ to have the dimension of kg.

5.2 Particle Affects Field.

In Eq.(5.4), the action integral of the field is presented.
The action integral for the particle is taken from Eq.(5.2) with $c = 1$.
At pages 163 and 164, Susskind defined the interaction Lagrangian:
$\mathcal{L}_{interaction} = -g\phi(t,x)[1 - \frac{\dot{x}^2}{2}]$.
Then, Susskind continued with a simple case: a particle at rest at $x = 0$.
At the top of page 165 the Action integral representing interaction is given, Eq.(5.6).

Note: I suppose $\int -[m + g\phi(t,0)]dt$, should read : $\int -g\phi(t,0)dt$.
Eq.(5.7): $\Rightarrow \mathcal{L}_{interaction} = -g\rho(x)\,\phi(t,x)$.
More about the Dirac δ-function: Susskind et al (1), and Dirac.
Just below Eq.(5.12) in the text: $g\,\phi(x) \Rightarrow g\phi(t,x)$.

5.2.1 Equations of Motion

With Eq.(5.14):
$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}_{total}}{\partial(\frac{\partial\phi}{\partial t})} \Rightarrow \frac{\partial\mathcal{L}_{total}}{\partial(\frac{\partial\phi}{\partial t})} = \frac{\partial\phi}{\partial t}$.
Next:
$\frac{\partial\mathcal{L}_{total}}{\partial(\frac{\partial\phi}{\partial x^1})} = \frac{\partial\mathcal{L}_{total}}{\partial(\frac{\partial\phi}{\partial x})} = -\frac{\partial\phi}{\partial x}$.
To conclude this Lecture: "*Fields and particles affect each other through a common term in the Lagrangian".*

5.2.2 Time Dependence

The equation demonstrating time dependence is presented at the bottom of page 172.

5.3 Upstairs and Downstairs Indices

On page 175 the metric $\eta_{\mu\nu}$ is introduced; a matrix.
μ represents the row of the matrix, ν the row.

Note: subscript indices $\mu\nu$ are subscript by convention?
With the metric and A^ν, A_μ is obtained: $\sum_\nu \eta_{\mu\nu} A^\nu = A_\mu$.
So, for $\mu = 0$, the first row,
$\sum_\nu \eta_{\mu\nu} A^\nu = A_\mu \Rightarrow -1 \cdot A^t = -A^t$, for $\mu = 0$, etc.

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} A^t \\ A^x \\ A^y \\ A^z \end{pmatrix} = \begin{pmatrix} -A^t \\ A^x \\ A^y \\ A^z \end{pmatrix}$$

5.4 Einstein Sum Convention

The convention is demonstrated with Eq. (5.19).
Note: At the bottom of page 177, Susskind writes: "*The operation of Expression 5.19* $[A^\mu A_\mu]$ *has the effect of changing the sig of the time component.*" I do not understand the summation changes the time component:

$$A^\mu A_\mu = (A^t\ A^x\ A^y\ A^z) \begin{pmatrix} -A^t \\ A^x \\ A^y \\ A^z \end{pmatrix}.$$

I understood the metric changes the sign of the time component.

Exercise 5.1 About the dummy index

Show that $A^\nu A_\nu$, has the same meaning as $A^\mu A_\mu$.
As mentioned by Susskind on the pages 177-178, "*An index that triggers the summation convention, like* ν, *does not have a specific value. It is called a summation or dummy index*".
"*If we replace* ν *with any other Greek letter, the expression would have exactly the same meaning*".

Exercise 5.2 About reversing the *upstairs* and *downstairs* index.

Write an expression that undoes the effect of Eq. (5.20). In other words, how do we "go backwards"?
Eq. (5.20): $A_\mu = \eta_{\mu\nu} A^\nu$. So, to undo this effect is means to find A^ν, given Eq. (5.20)?
We know how the metric operates on a 4-vector. Repeat this operation with $\eta_{\mu\nu}$. Well, that is equal to multiply with the unity matrix. How to write this down?
Is it something like Susskind wrote on page 177 for $\dot{X}^i$?
Hence,
$\eta^{\mu\nu}\eta_{\mu\nu} A^\nu = \eta^{\mu\nu} A_\mu = A^\nu$.
So,

$$\eta^{\mu\nu}\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This is an educated guess, since I assumed $\eta^{\mu\nu} = \eta_{\mu\nu}$.
Note: $\eta^{\mu\nu}\eta_{\mu\nu} A^\nu$, am I allowed to do this? Three ν's! Or should I consider this as a two-step procedure? First $\eta_{\mu\nu} A^\nu = A_\mu$, and then $\eta^{\mu\nu} A_\mu = A^\nu$?

With respect to my note above, I assumed Susskind explained this in more detail at the top of page 179
Note:
At the middle of page 179: "*If* A^μ *happens to be a displacement such as* X^μ, *then it is the*

same as the quantity τ^2, except with an overall minus sign; in other words, it is $-\tau^2$".
Well, what does this mean?
$\tau^2 = -(X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2$.
With this expression: " *If A^μ happens to be a displacement such as X^μ* … … ".
So, with the above expression:
$\tau^2 = -(X^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2$. Consequently, I do not understand the expression at the middle of page 179.

A contraction exercise at the bottom of page 180

Proof $A^\mu B_\mu = A_\mu B^\mu$.
Assume this expression to be correct.
Then,
$A^\mu \eta_{\mu\nu} B^\nu = A_\mu B^\mu$.
In the expression on the left-hand side we have $A^\mu \eta_{\mu\nu}$.
After contraction $A^\mu \eta_{\mu\nu} = A_\nu$.
Hence,
$A^\mu B_\mu = A^\mu \eta_{\mu\nu} B^\nu = A_\nu B^\nu = A_\mu B^\mu$.
since the summation is over dummy index $\Rightarrow A^\mu B_\mu = A_\mu B^\mu$.
End of *Proof*

5.5 Scalar Field Conventions.

A theorem:
Given the product $A_\mu B^\mu$ to be a scalar, and A_μ to be a 4-vector $\Rightarrow$ B^μ is a 4-vector.
Proof:
Suppose B^μ to be no 4-vector. Then, B^μ is a scalar.
Consequently $A_\mu B^\mu$ is not a scalar. This contradicts the assumption.
Hence B^μ is a 4-vector.
End of Proof.

Note: with the contravariant transformations used A_μ and the presumption B^μ to be a -4 vector, the presumption in the proof when using these transformations, is not contradicted. However, I assumed in addition the transformation of a product is equal to the product of transformations.

Next, Susskind considers the in value of the scalar $\phi(x)$, pages 181-182. There, the theorem mentioned above is applied.
Question: why not present the covariant 4-vector ∂X^μ as ∂X_μ?
At the bottom of page 182, Susskind summarizes: *The derivatives of a scalar with respect to X^μ form a covariant vector, written with the shorthand symbol: $\partial_\mu \phi$.*
Note: in Eq. (4.21) we met this symbol before. In addition, I prefer, in Eq. (5.24), $\frac{\partial \phi(t,x)}{\partial X^\mu}$.
Since dX^μ is a 4-vector , $\frac{\partial}{\partial t}$ included
On page 183, the contravariant version $\partial^\mu \phi$ is presented.

5.6 A New Scalar

The new scalar is $\partial^\mu \phi \partial_\mu \phi$.

Susskind: *This makes it easy to see that the Lagrangian for that scalar field is itself a scalar*.

5.7 Transforming Covariant Components.

In this Lecture the formulas for transforming covariant components are presented.

I used these formulas for an additional proof of the theorem given in Lecture 5.5:

$A_\mu B^\mu = (A_\mu B^\mu)' = A_\mu{}' B^{\mu'}$.

5.8 Mathematical Interlude: Using Exponentials to solve Wave Equations.

$\phi(x,t) = e^{i(kx-\omega t)}$ is the main building block for wave equations.

5.9 Waves

The Lagrangian is given in Eq. (5.26).

Page 188:

$\frac{d}{dt}\frac{\partial \frac{1}{2}(\frac{\partial \phi}{\partial t})^2}{\partial \frac{\partial \phi}{\partial t}} = \frac{d}{dt}\frac{\partial \phi}{\partial t} = \frac{\partial^2 \phi}{\partial t^2}$, etc.

The differential equation of motion for $\phi(x,t)$: Eq. (5.27). This equation we met in Lecture 4.4.4-Eq. (4.25).

Susskind made some remarks on the Klein-Gordon equation. Does this equation come close to the Dirac equation, page 253?

Interlude: Crazy Units

I.1 Units and Scales

I.2 Planck Units

I.3 Electromagnetic Units

"The force between two coulomb-size charges is enormous. To account for this, we must put a huge constant into the force law. Instead of

$F = \frac{q_1 q_2}{4\pi r^2}$,

we write

$F = \frac{q_1 q_2}{4\pi \epsilon_0 r^2}$, *Eq (I.4),*

where ϵ_0 *is the small number* 8.85×10^{-12}. *"*

Question: which force is enormous? Which constant is huge?

Lecture 6 The Lorentz Force Law

Page 208: "….. *electric and magnetic forces transform into each other under Lorentz transformation".*

The Lorentz law is presented in Eq. (6.1).

6.1 Extending Our Notation

The basic building blocks are 4-vectors.

6.1.1 4-Vector Summary.

Here, the results of Lecture 5 are summarized.
For the switch from contravariant notation to covariant notation the metric $\eta_{\mu\nu}$ is used, summarized in Eq. (6.2).

6.1.2 Forming Scalars.

Formula: $A_\nu B^\mu$ ⇒scalar.

6.1.3 Derivatives

In Lecture 5, page 182, is explained why the derivatives are the covariant components of a 4-vector. This is summarized in Eq. (5.24).

- $\partial_\mu \phi = \frac{\partial \phi}{\partial X^\mu}$, is a covariant 4-vector, with ϕ a scalar,
- $\partial_\mu B^\mu(t, x)$, is a scalar, with $B^\mu(t, x)$ a 4-vector.

6.1.4 General Lorentz Transformation

An important statement: *Physics is invariant under simple Lorentz transformations and under a broader category of transformations such as rotations of space.*
Lorentz transformations are presented as a matrix operation, Eq. (6.3).
The matrix is denoted by $L^\mu{}_\nu$. This operator is used with contravariant 4-vectors.
What is the difference between $L^\mu{}_\nu$ and $L_\nu{}^\mu$? Does the latter not exist?
In Eq. (6.4), the operation is demonstrated.
What about the matrix for a transformation along the y-axis, mentioned on page 217?
I think ,I shuffled the matrix elements in the right way, with the following result:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & 0 & \frac{-v}{\sqrt{1-v^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-v}{\sqrt{1-v^2}} & 0 & \frac{1}{\sqrt{1-v^2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}.$$

Next, a rotation in the y-z plane is analysed. In Eq. (6.5), the rotation matrix is presented illustrated by Figure 2.

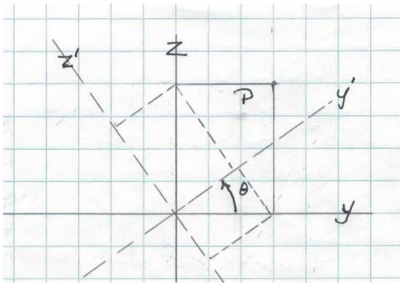


Figure 2 The rotation matrix Eqs. (6.5) and (6.6)

6.1.5 Covariant Transformations

With the equations presented in Lecture 5.7, page 185, the transformation matrix along the x-axis can be constructed:

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} . \qquad \text{(C.6.1.5.1)}$$

The difference with Eq. (6.4) is demonstrated.
On page 219 the matrix ${M_\mu}^\nu$, is introduced.
So, may I assume, the way the indices are positioned $L^\mu{}_\nu$, given on the foregoing page, $L^\mu{}_\nu$is used for contravariant transformations? The way it is formulated on page 219, does not clarify it sufficiently for me.
However I assume $L^\mu{}_\nu$ to be used for contravariant transformations and ${M_\mu}^\nu$ for covariant transformations.

The matrix to transform covariant 4-vectors is denoted ${M_\mu}^\nu$. Since the matrix $L^\mu{}_\nu$, Eq. (6.3), and ${M_\mu}^\nu$, Eq. (6.7), *represent the same physical transformation between coordinate frames, these matrices must be connected.*
Hence,
$M = \eta L \eta$.
Susskind let the proof of this expression to the reader. A question came to my mind: where are the sub- and superscripts?
The relation can be demonstrated by the metric to the left and to the right on the contravariant matrix presented in Eq. (6.4). Then the expression (C.6.1.5.1) is obtained. I applied matrix multiplication. This is certainly not a proof. In Exercise (6.1), I show the details of this multiplication.
At the bottom of page 219:
$\eta^{-1} = \eta$.
So,
$M = \eta L \eta \Longrightarrow \eta M \eta = L$.

Exercise 6.1 About contravariant and covariant transformations.

Given the transformation equation, Eq. (6.3), $(A')^\mu = L^\mu{}_\nu A^\nu$, for the contravariant components of a 4-vector A^ν , where $L^\mu{}_\nu$ is a Lorentz transformation matrix, show that the Lorentz transformation for A's covariant components is
$(A')_\mu = {M_\mu}^\nu A_\nu$,
where
$M = \eta L \eta$.
In Eq.(6.3), the contravariant transformation is presented:
$(A')^\mu = L^\mu{}_\nu A^\nu$.
Show he covariant transformation to be:
$(A')_\mu = {M_\mu}^\nu A_\nu$.
So, I have to show
$(A')_\mu = \eta L \eta A_\nu$.
Remark: $\eta L \eta$ is about matrix multiplication.

In Lecture 5.7, the covariant transformation along the x-axis is given. Then, the matrix ${M_\mu}^\nu$ is, (C.6.1.5.1):

$$ {M_\mu}^\nu = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

The matrix for the contravariant transformation, ${L^\mu}_\nu$, along the x-axis is given in Eq.(6.4):

$$ {L^\mu}_\nu = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

So, with $M = \eta L \eta$ and metric matrices:

$$ {M_\mu}^\nu = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = $$

$$ = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. $$

End of Exercise.

6.2 Tensors

Scalars and vectors are example of tensors.

6.2.1 Rank 2 Tensors

On page 221, Susskind presented an answer to a question, at least for a contravariant tensor of rank 2.: The transformation of the product of two 4-vectors equals the product of the transformation of the 4-vectors.

6.2.2 Tensors of Higher Rank

The generalisation for higher rank tensors is presented in Eq.6.10).
Tensors are defined or determined by the transformation properties.

6.2.3 Invariance of Tensor Equations

"The real power behind tensor notation is that tensor equations are frame invariant."

6.2.4 Raising and Lowering Indices

Here Susskind explained the meaning of *other variants* of a tensor.
The metric comes into play,
Page 224: ${T^\mu}_\nu = T^{\mu\sigma}\eta_{\sigma\nu}$.
On pages 224 and 225, Susskind showed an easy way about raising and lowering operators by showing the effect on the time index and the space index.

To conclude: "*Whenever you lower or raise a time component, you change sign. That is all there is.*"

6.2.5 Symmetric and Antisymmetric Tensors

Susskind: *"In general $T^{\mu\nu} \neq T^{\nu\mu}$."*
To make it a bit clearer: $A^0B^1 \neq A^1B^0$.
An example of a tensor invariant to changing the order of indices, is a symmetric tensor. On page 226 an example is presented.
On page 227, the antisymmetric tensor is introduced. An example of an antisymmetric tensor:
$A^\mu B^\nu - A^\nu B^\mu \Rightarrow A^\nu B^\mu - A^\mu B^\nu = -(A^\mu B^\nu - A^\nu B^\mu)$, defined at the top of page 227.

6.2.6 An Antisymmetric Tensor.

In Eq. (6.11), the covariant components of the antisymmetric tensor $F_{\mu\nu}$ are given. The order of appearance is not explained. How to find Eq. (6.11)?

6.3 Electromagnetic Fields

The equation of motion of a charged particle in electromagnetic field is presented. The equation of motion is based on the Lorentz force law: Eq. (6.12).

6.3.1 The Action Integral and the Vector Potential

Just below the middle of page 230:
"*How can we modify this Lagrangian to describe the effects of the electromagnetic field?*"
I suppose "*This Lagrangian*" , is presented in Eq. (4.29):
$-[m + \phi(t,x)]\sqrt{1 - v^2}$, or is it Eq. (4.27)?
The question to be worked upon is: "*How can we use $A_\mu(t,x)$, a 4-vector, to construct an action for a particle in an electromagnetic Field?*"
Eq. (6.13):
The free particle action, the first integral in Eq. (6.13), is added to the action for a particle moving in an electromagnetic field, the second integral in Eq. (6.13).

6.3.2 The Lagrangian

In Eq. (6.18) The Lagrangian for the free particle and the electromagnetic field is given.

6.3.3 Euler-Lagrange Equations

Eqs (6.18) and(6.19):
- $\frac{\partial}{\partial \dot{X}^p} eA_0(t,x) = 0$,
- $\frac{\partial}{\partial \dot{X}^p} e\dot{X}^p A_p(t,x) = eA_p(t,x)$, 3 p's,
- $\frac{\partial}{\partial \dot{X}^p}\left(-m\sqrt{1-\dot{x}^2}\right) = m\frac{\dot{X}^p}{\sqrt{1-\dot{x}^2}}$,

where use has been made of $\dot{x}^2 = \dot{X}^p\dot{X}_p$.
With these ingredients, Eq. (6.20) is obtained.

In Eq. (6.20), I observe one index: a covariant expression.
The remark below Eq.6.20): "*This makes no difference because space components such a p can be moved upstairs or downstairs at will without changing their value*". This remark is about a Cartesian frame?

On the second halve of page 236, Susskind looked into the space derivative of $\mathcal{L}$:
$\frac{\partial \mathcal{L}}{\partial X_p}$, a covariant expression.
The last term analysed raises a question, $e\dot{X}^p A_p(t,x)$, Eq. (6.18):
$\frac{\partial}{\partial X^p}[e\dot{X}^p A_p(t,x)]$,
A_p becomes A_n. I suppose this not to be an arbitrary change of the index? Is it about the summation convention or is something else at play?
Most probably, the answer is found in the equation at the top of page 237:
$\frac{\partial \mathcal{L}}{\partial X^p} = e\frac{\partial A_0}{\partial X^p} + e\dot{X}^p \frac{\partial A_n(t,x)}{\partial X^p}$.
On the left-hand side of this expression has one index p. On the right-hand side, $e\frac{\partial A_0}{\partial X^p}$, there is again one index p (*nonsummed-free covariant index*, Susskind). The other term on the right-hand side, $e\dot{X}^p \frac{\partial A_n(t,x)}{\partial X^p}$, a summation over p (contraction). This latter expression cannot be correct. After the contraction, no p is available. I suppose, an educated guess, we need to keep ∂X^p. Now, which index is available for $\dot{X}^p$? The only possible answer here is: A_n and $\dot{X}^p$ must have the same index $\neq p$. My conclusion is based upon the first term on the right-hand side of Eq. (6.21): $e\frac{\partial A_0}{\partial X^p}$.
So, $e\dot{X}^p \frac{\partial A_n(t,x)}{\partial X^p} \Rightarrow e\dot{X}^n \frac{\partial A_n(t,x)}{\partial X^p}$.
Then, the well-formed Eq. (6.21) is obtained.
On page 238, the time derivative of
$eA_p(t,x)$ is analysed.
First, $e\frac{\partial A_p(t,x)}{\partial t}$ is given.
Since the position is a function of time, there is also:
$e\frac{\partial A_p(t,\vec{x})}{\partial \vec{x}} = e\left(\frac{\partial A_p}{\partial x}\frac{dx}{dt} + \frac{\partial A_p}{\partial y}\frac{dy}{dt} + \frac{\partial A_p}{\partial z}\frac{dz}{dt}\right)$.

Note: in total we have $e\left(\frac{\partial A_p(t,x)}{\partial t} + \frac{\partial A_p}{\partial x}\frac{dx}{dt} + \frac{\partial A_p}{\partial y}\frac{dy}{dt} + \frac{\partial A_p}{\partial z}\frac{dz}{dt}\right)$. In fluid dynamics this is called the total derivative.

So, we have for the time derivative:
$e\frac{\partial A_p(t,x)}{\partial t} + e\frac{\partial A_p(t,x)}{\partial X^n}\dot{X}^n$.
Finally, Eq.(6.22) is obtained.
Susskind: "*This equation is well formed because each side has a summation index and a free covariant index* p".
Eq. (6.22) is analysed resulting into Eq. (6.24)

Exercise 6.2 About the cross products in expression (6.28)

Expression (6.28) was derived by identifying the index p with the z component of space, and then summing over n for the values (1, 2, 3). Why does not Expression (6.28) contain a v_z term?
Let us find out what the result could be adding v_z to Exp. (6.28).
Start with Exp. (6.27):

$\dot{X}^n\left(\frac{\partial A_n(t,x)}{\partial X^p}-\frac{\partial A_p(t,x)}{\partial X^n}\right)\Rightarrow v_x\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+v_y\left(\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}\right)+v_z\left(\frac{\partial A_z}{\partial z}-\frac{\partial A_z}{\partial z}\right).$

Hence

$\dot{X}^n\left(\frac{\partial A_n(t,x)}{\partial X^p}-\frac{\partial A_p(t,x)}{\partial X^n}\right)\ \Rightarrow v_x\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+v_y\left(\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}\right).$

End of Exercise.

Page 242:

Compare Eq. (6.26) with Exp. (6.28)⇒ $-v_yB_x=v_y\left(\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}\right)\Rightarrow B_x=-\left(\frac{\partial A_y}{\partial z}-\frac{\partial A_z}{\partial y}\right).$

In Eq. (6.31), the Lorentz invariant form of the equation of motion is presented.

6.3.4 Lorentz Invariant Equations.

Eq. (6.31) will be written in a 4-dimensional form with covariant and contravariant notation.
The starting point is Eq. (6.24).
Use Eq. (2.16):

$U^\mu=\frac{dX^\mu}{d\tau},$

and Eq.(3.9),

$U^i=\frac{v^i}{\sqrt{1-v^2}}=\frac{\dot{x}^p}{\sqrt{1-\dot{x}^2}}=\frac{dX^p}{d\tau}.$

Then, Susskind , on page 244 just below Eq.(6.32), mentioned p to be a Latin index. It is about space coordinates and the time derivative of these space coordinates in a Cartesian frame. Consequently, downstairs, or upstairs does not matter.
However, the equation of motion must be brought into 4 vector form: the relative acceleration. It is not just multiplying the acceleration term on the left-hand side of Eq. (6.24) by $\frac{dt}{d\tau}$, both sides of Eq. (6.24) need to be multiplied by $\frac{dt}{d\tau}$.
Having done this, the equation of motion can be written in a more elegant form, considering the proper use of indices, Eq. (6.33).

6.3.5 Equations with 4-Velocity

The equation of motion is rewritten using the 4-velocity ⇒ Eq. (6.34).
In addition, the spatial part of the equation of motion is presented in Eq. (6.35).

6.3.6 Relationship of Aμ to $\vec{E}$ and $\vec{B}$

In this Lecture, Susskind summarized the relations between the vector potential A_μ , $\vec{E}$, and $\vec{B}$.

6.3.7 The Meaning of U^μ

Susskind evaluates the zeroth equation of (6.34).
Eq. (6.34):

$m\frac{dU_\mu}{d\tau}=e\left(\frac{\partial A_\nu}{\partial X^\mu}-\frac{\partial A_\mu}{\partial X^\nu}\right)U^\nu.$

With $\mu=0$ in Eq.(6.34) and contraction over ν :

$m\frac{dU_0}{dt}=e\left(\frac{\partial A_0}{\partial X^0}-\frac{\partial A_0}{\partial X^0}\right)U^0+e\left(\frac{\partial A_1}{\partial X^0}-\frac{\partial A_0}{\partial X^1}\right)U^1+\cdots+\ e\left(\frac{\partial A_3}{\partial X^0}-\frac{\partial A_0}{\partial X^3}\right)U^3\Rightarrow$

$\Rightarrow\ m\frac{dU_0}{dt}=e\left(\frac{\partial A_n}{\partial X^0}-\frac{\partial A_0}{\partial X^n}\right)\frac{dX^n}{dt}$, Eq.(6.37).

Note: $\tau\Rightarrow t$, on both sides.

At the bottom of page 248 and top of page 249, relativistic kinetic energy is recalled. Remark: Why the Eqs. (3.36) and (3.37) are mentioned here, I do not know. At the bottom of page 249, "*So we see that the fourth equation*…….". Fourth equation?

6.4 Interlude on the Field Tensor

Eq.(6.11): $F_{\mu\nu}$.

With Eq.(6.38) or (6.39) ⇒ $F_{10} = \frac{\partial A_0}{\partial X^1} - \frac{\partial A_1}{\partial X^0} = \frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t} = E_x$, Eq.(6.36).

Then,

$F_{01} = -F_{10} = -\frac{\partial A_0}{\partial X^1} + \frac{\partial A_1}{\partial X^0} = -E_x$, e.g., the first row in Eq.(6.41).

What about $F^{\mu}{}_{\nu}$? I expect the only effect on the index, sub or super, 0 . Representing the time component: a change of sign.

So, there is no difference between F_{10} and $F^{1}{}_{0}$, both represent E_x.

There is a difference between F_{01} and $F^{0}{}_{1}$: $-E_x \Rightarrow E_x$.

Thinking of and applying the metric, pages 175 and 176:

$\eta_{\mu\nu}F_{\mu\nu} = F^{\mu}{}_{\nu}$ and $\eta_{\mu\nu}F^{\mu\nu} = F_{\mu}{}^{\nu}$. I suppose this to be correct.

Lecture 7 Fundamental Principles and Gauge Invariance.

Four Principles govern physical laws:

- The action Principle
- Locality
- Lorentz invariance
- Gauge invariance.

7.1 Summary of Principles

Action Principle

Derived from the action Principle:

- Conservation of energy
- Conservation of momentum
- Relation between conservation laws and symmetries

Locality

Short time effect is local

Lorentz Invariance

$\mathcal{L} = Scalar$

Gauge Invariance

Susskind: *"… gauge invariance has to do with changes that you can make to the vector potential without affecting the physics".*

7.2 Gauge In variance

An invariance ⇔ a symmetry.

7.2.1 Symmetry examples

Page 261: "*Adding a constant to the field makes no difference; it will still minimize the action ⇒ adding a constant to such a field is asymmetry, or an invariance*".

7.2.2 A New Kind of Invariance

Gauge transformation ⇒ gauge invariance: *"The vector potential, A_μ, can be changed, in a certain way, without any effect on the behavior of the charged particle".*

7.2.3 Equation of Motion

A concise notation for the Lorentz force law is given, page 265:

$$m\frac{d^2X_\mu}{d\tau^2} = eF_{\mu\nu}U^\nu .$$

Reformulate the Gauge invariance: "*Any change to the vector potential that does not affect the field tensor $F_{\mu\nu}$, will not affect the motion of the particle*".

In Eq.(7.7) a gradient of a scalar is added to the vector potential.

Eq.(7.8) demonstrates the statement of the invariance.

Keep in mind, in Eq. (7.7) the components of the gradient of the scalar are added to the components of the vector potential ⇒ the reason for the different indices:

$\frac{\partial S}{\partial X^\nu}$ and $\frac{\partial S}{\partial X^\mu}$.

7.2.4 Perspective

Susskind reflects on the usefulness of adding a scalar to the vector potential⟹by using all possible scalars all the properties of a theory can be observed.

Lecture 8 Maxwell's Equations

Reference is made to Einstein's paper on the subject matter.

8.1 Einstein's Example

In Fig's 8.1 and 8.2 Einstein's example is illustrated from the point of view of a moving particle and the point of view of a moving magnet.

Bottom page 273: "*By this simple thought experiment Einstein derived the fact that magnetic fields must transform into electric fields under Lorentz transformation*".

8.1.1 Transforming the Field Tensor

The field tensor in the rest frame:

$F_{\mu\nu}$, Eq.(6.41).

What are the components of the field tensor in the moving frame?

On page 274, Susskind summarized the Lorentz transformation of 4-vectors.

In Lecture 6 the Lorentz matrix $L^\mu{}_\nu$ represents the Lorentz transformation:

$(X')^\mu = L^\mu{}_\nu X^\nu$, Eq.(8.1).

To find out about the transformation of the field tensor, $F^{\mu\nu}$ Eq.(6.42) is used instead of $F_{\mu\nu}$.

How does $F^{\mu\nu}$ transform? This is defined in Eq.(8.2).

The job to be done is to compute the y component, $(E')^y$ in the primed(moving) frame.

In terms of the field tensor, $(E')^y$ being a component of the field tensor can be written as:

$+(E')^y \equiv (F')^{0y}$ ⟹ the zeroth row and the second, y, column.

For completeness, I will use Eq.(8.2):

$(F')^{\mu\nu} = L^\mu{}_\sigma L^\nu{}_\tau F^{\sigma\tau}$, $\sigma, \tau = 0,1,2,3.$

For the Einstein's thought experiment, we know the only contributions in the unprimed field tensor are:

$F^{12}(=B_z)$, and $F^{21}(=-B_z)$.
So,
$(F')^{02}=L^0{}_1L^2{}_2\,F^{12}+L^0{}_2L^2{}_1\,F^{21}=\frac{-v}{\sqrt{1-v^2}}\cdot 1\cdot B_z+0=\frac{-vB_z}{\sqrt{1-v^2}}$, Eq.(8.4).
Susskind: *"What do we know about the original unprimed field tensor $F^{\mu\nu}$ as it applies to Einstein's example?"*, bottom of page 276. So, I think it is about Fig.8.1.

Remark: Susskind used for the element of the field tensor the notation F^{xy}. The x^{th} row and the y^{th} column$\Longrightarrow +B_z$. That's ok. However, it becomes a bit confusing in Eq. (8.3)$\Longrightarrow$three y indices. Which to contract? In Eq. (8.2), the transformation of a tensor, there are no three equal indices. There it is clear how to contract.

On page 277, the result of the transformation is summarized, Eq. (8.4), a single component of the primed field tensor.

Exercise 8.1: Transformation of the field tensor for a charge at rest and a moving observer

Consider an electric charge at rest, with no additional electric or magnetic fields present. In terms of the rest frame components, (E_x,E_y,E_z), what is the x component of the electric field for an observer moving in the negative x direction with velocity v? What are the y and z components? What are the corresponding components of the magnetic field?
At first sight, it looks like Einstein's thought experiment as illustrated in Fig.8.2. However, "..*with no additional electric or magnetic fields present*". What does this mean? Just a moving observer and no magnet? There are rest frame components (E_x,E_y,E_z). It is of no importance where this field originates from?
I consider the rest frame to be the unprimed frame. With a moving observer and no magnet, the field tensor is:

$$F^{\mu\nu}=\begin{pmatrix}0 & +E_x & +E_y & +E_z\\ -E_x & 0 & 0 & 0\\ -E_y & 0 & 0 & 0\\ -E_z & 0 & 0 & 0\end{pmatrix}.$$

The Lorentz matrix:

$$L^\mu{}_\nu=\begin{pmatrix}\frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} & 0 & 0\\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Now, I look into the application of the two-index transformation: Eq.(8.2).
$(F')^{\mu\nu}=L^\mu{}_\sigma L^\nu{}_\tau F^{\sigma\tau}$, $\sigma,\tau=0,1,2,3$.
So, calculating $(F')^{\mu\nu}$, there are 16 elements to be analysed. Four of them are zero: the diagonal elements of the field tensor.
Note: I do not use the anti-symmetric of the field tensor. In this way a have a check on the results.
Let us find out about the first row and the second column of $(F')^{\mu\nu}$.
$(F')^{01}\equiv(F')^{0x}=L^0{}_0L^x{}_0F^{00}+L^0{}_1L^x{}_0F^{10}+L^0{}_2L^x{}_0F^{20}+L^0{}_3L^x{}_0F^{30}+$
$+L^0{}_0L^x{}_1F^{01}+L^0{}_1L^x{}_1F^{11}+L^0{}_2L^x{}_1F^{21}+L^0{}_3L^x{}_1F^{31}+$

$+L^0{}_0 L^x{}_2 F^{02} + L^0{}_1 L^x{}_2 F^{12} + L^0{}_2 L^x{}_2 F^{22} + L^0{}_3 L^x{}_2 F^{32} +$
$+L^0{}_0 L^x{}_3 F^{03} + L^0{}_1 L^x{}_3 F^{13} + L^0{}_2 L^x{}_3 F^{23} + L^0{}_3 L^x{}_3 F^{33}.$

Due to the absence of the magnetic field, and use the diagonal in the field tensor, $(F')^{01} \equiv (F')^{0x}$ reduces into:

$(F')^{01} \equiv (F')^{0x} = L^0{}_1 L^x{}_0 F^{10} + L^0{}_2 L^x{}_0 F^{20} + L^0{}_3 L^x{}_0 F^{30} + L^0{}_0 L^x{}_1 F^{01} + L^0{}_0 L^x{}_2 F^{02} +$
$+L^0{}_0 L^x{}_3 F^{03}.$

Next, I use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{01} \equiv (F')^{0x} = L^0{}_1 L^x{}_0 F^{10} + L^0{}_0 L^x{}_1 F^{01} = \frac{-v}{\sqrt{1-v^2}} \frac{-v}{\sqrt{1-v^2}} (-E_x) + \frac{1}{\sqrt{1-v^2}} \frac{1}{\sqrt{1-v^2}} E_x = E_x .$

Hence,

$(F')^{01} \equiv (F')^{0x} = E_x .$

A result to be expected?

What about $(F')^{02} \equiv (F')^{0y}$?

$(F')^{02} \equiv (F')^{0y} = L^0{}_0 L^y{}_0 F^{00} + L^0{}_1 L^y{}_0 F^{10} + L^0{}_2 L^y{}_0 F^{20} + L^0{}_3 L^y{}_0 F^{30} +$
$+L^0{}_0 L^y{}_1 F^{01} + L^0{}_1 L^y{}_1 F^{11} + L^0{}_2 L^y{}_1 F^{21} + L^0{}_3 L^y{}_1 F^{31} +$
$+L^0{}_0 L^y{}_2 F^{02} + L^0{}_1 L^y{}_2 F^{12} + L^0{}_2 L^y{}_2 F^{22} + L^0{}_3 L^y{}_2 F^{32} +$
$+L^0{}_0 L^y{}_3 F^{03} + L^0{}_1 L^y{}_3 F^{13} + L^0{}_2 L^y{}_3 F^{23} + L^0{}_3 L^y{}_3 F^{33}.$

Use the absence of a magnetic field in the field tensor:

$(F')^{02} \equiv (F')^{0y} = L^0{}_1 L^y{}_0 F^{10} + L^0{}_2 L^y{}_0 F^{20} + L^0{}_3 L^y{}_0 F^{30} + L^0{}_0 L^y{}_1 F^{01} + L^0{}_0 L^y{}_2 F^{02} +$
$+L^0{}_0 L^y{}_3 F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{02} \equiv (F')^{0y} = L^0{}_0 L^y{}_2 F^{02} = \frac{1}{\sqrt{1-v^2}} E_y .$

A result to be expected?

Now, $(F')^{03} \equiv (F')^{0z}$:

$(F')^{03} \equiv (F')^{0z} = L^0{}_0 L^z{}_0 F^{00} + L^0{}_1 L^z{}_0 F^{10} + L^0{}_2 L^z{}_0 F^{20} + L^0{}_3 L^z{}_0 F^{30} +$
$+L^0{}_0 L^z{}_1 F^{01} + L^0{}_1 L^z{}_1 F^{11} + L^0{}_2 L^z{}_1 F^{21} + L^0{}_3 L^z{}_1 F^{31} +$
$+L^0{}_0 L^z{}_2 F^{02} + L^0{}_1 L^z{}_2 F^{12} + L^0{}_2 L^z{}_2 F^{22} + L^0{}_3 L^z{}_2 F^{32} +$
$+L^0{}_0 L^z{}_3 F^{03} + L^0{}_1 L^z{}_3 F^{13} + L^0{}_2 L^z{}_3 F^{23} + L^0{}_3 L^z{}_3 F^{33}.$

Use the absence of a magnetic field in the field tensor:

$(F')^{03} \equiv (F')^{0z} = L^0{}_1 L^z{}_0 F^{10} + L^0{}_2 L^z{}_0 F^{20} + L^0{}_3 L^z{}_0 F^{30} + L^0{}_0 L^z{}_1 F^{01} + L^0{}_0 L^z{}_2 F^{02} +$
$+L^0{}_0 L^z{}_3 F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{03} \equiv (F')^{0z} = \frac{1}{\sqrt{1-v^2}} E_z .$

For the moving observer, the electric field is:

$E' = (E_x, \frac{1}{\sqrt{1-v^2}} E_y, \frac{1}{\sqrt{1-v^2}} E_z).$

The magnetic field components, $F^{23} = B_x = 0$:

$(F')^{23} = L^2{}_0 L^3{}_0 F^{00} + L^2{}_1 L^3{}_0 F^{10} + L^2{}_2 L^3{}_0 F^{20} + L^2{}_3 L^3{}_0 F^{30} +$
$+L^2{}_0 L^3{}_1 F^{01} + L^2{}_1 L^3{}_1 F^{11} + L^2{}_2 L^3{}_1 F^{21} + L^2{}_3 L^3{}_1 F^{31} +$
$+L^2{}_0 L^3{}_2 F^{02} + L^2{}_1 L^3{}_2 F^{12} + L^2{}_2 L^3{}_2 F^{22} + L^2{}_3 L^3{}_2 F^{32} +$
$+L^2{}_0 L^3{}_3 F^{03} + L^2{}_1 L^3{}_3 F^{13} + L^2{}_2 L^3{}_3 F^{23} + L^2{}_3 L^3{}_3 F^{33}.$

Use the absence of a magnetic field in the unprimed field tensor:

$(F')^{23} = L^2{}_1 L^3{}_0 F^{10} + L^2{}_2 L^3{}_0 F^{20} + L^2{}_3 L^3{}_0 F^{30} + L^2{}_0 L^3{}_1 F^{01} + L^2{}_0 L^3{}_2 F^{02} +$
$+L^2{}_0 L^3{}_3 F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{23} = 0.$

Another one $(F')^{12}$, I start using the absence of the magnetic field elements in the unprimed field tensor.

In general for this exercise:

$(F')^{\mu\nu} = L^{\mu}{}_{1}L^{\nu}{}_{0}F^{10} + L^{\mu}{}_{2}L^{\nu}{}_{0}F^{20} + L^{\mu}{}_{3}L^{\nu}{}_{0}F^{30} + L^{\mu}{}_{0}L^{\nu}{}_{1}F^{01} + L^{\mu}{}_{0}L^{\nu}{}_{2}F^{02} + $
$+L^{\mu}{}_{0}L^{\nu}{}_{3}F^{03}.$

$(F')^{12} = L^{1}{}_{1}L^{2}{}_{0}F^{10} + L^{1}{}_{2}L^{2}{}_{0}F^{20} + L^{1}{}_{3}L^{2}{}_{0}F^{30} + L^{1}{}_{0}L^{2}{}_{1}F^{01} + L^{1}{}_{0}L^{2}{}_{2}F^{02} + $
$+L^{1}{}_{0}L^{2}{}_{3}F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{12} = L^{1}{}_{0}L^{2}{}_{2}F^{02} = \frac{-v}{\sqrt{1-v^2}}E_y\ .$

The next element $(F')^{13}$, using the absence of the magnetic field elements in the unprimed tensor:

$(F')^{13} = L^{1}{}_{1}L^{3}{}_{0}F^{10} + L^{1}{}_{2}L^{3}{}_{0}F^{20} + L^{1}{}_{3}L^{3}{}_{0}F^{30} + L^{1}{}_{0}L^{3}{}_{1}F^{01} + L^{1}{}_{0}L^{3}{}_{2}F^{02} + $
$+L^{1}{}_{0}L^{3}{}_{3}F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{13} = L^{1}{}_{0}L^{3}{}_{3}F^{03} = \frac{-v}{\sqrt{1-v^2}}E_z.$

Next:

$(F')^{31} = L^{3}{}_{1}L^{1}{}_{0}F^{10} + L^{3}{}_{2}L^{1}{}_{0}F^{20} + L^{3}{}_{3}L^{1}{}_{0}F^{30} + L^{3}{}_{0}L^{1}{}_{1}F^{01} + L^{3}{}_{0}L^{1}{}_{2}F^{02} + $
$+L^{3}{}_{0}L^{1}{}_{3}F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{31} = L^{3}{}_{3}L^{1}{}_{0}F^{30} = \frac{v}{\sqrt{1-v^2}}E_z.$

Note: this indicates the anti-symmetric character of $(F')^{\mu\nu}$.

Let us find out.

$(F')^{10} = L^{1}{}_{1}L^{0}{}_{0}F^{10} + L^{1}{}_{2}L^{0}{}_{0}F^{20} + L^{1}{}_{3}L^{0}{}_{0}F^{30} + L^{1}{}_{0}L^{0}{}_{1}F^{01} + L^{1}{}_{0}L^{0}{}_{2}F^{02} + $
$+L^{1}{}_{0}L^{0}{}_{3}F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{10} = L^{1}{}_{1}L^{0}{}_{0}F^{10} + L^{1}{}_{0}L^{0}{}_{1}F^{01} = -\frac{1-v^2}{1-v^2}E_x = -E_x.$

The next element:

$(F')^{20} = L^{2}{}_{1}L^{0}{}_{0}F^{10} + L^{2}{}_{2}L^{0}{}_{0}F^{20} + L^{2}{}_{3}L^{0}{}_{0}F^{30} + L^{2}{}_{0}L^{0}{}_{1}F^{01} + L^{2}{}_{0}L^{0}{}_{2}F^{02} + $
$+L^{2}{}_{0}L^{0}{}_{3}F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{20} = L^{2}{}_{2}L^{0}{}_{0}F^{20} = \frac{-E_y}{\sqrt{1-v^2}}\ .$

Then,

$(F')^{30} = L^{3}{}_{1}L^{0}{}_{0}F^{10} + L^{3}{}_{2}L^{0}{}_{0}F^{20} + L^{3}{}_{3}L^{0}{}_{0}F^{30} + L^{3}{}_{0}L^{0}{}_{1}F^{01} + L^{3}{}_{0}L^{0}{}_{2}F^{02} + $
$+L^{3}{}_{0}L^{0}{}_{3}F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{30} = L^{3}{}_{3}L^{0}{}_{0}F^{30} = \frac{-E_z}{\sqrt{1-v^2}}\ .$

Is $(F')^{00}$, zero indeed?

$(F')^{00} = L^{0}{}_{1}L^{0}{}_{0}F^{10} + L^{0}{}_{2}L^{0}{}_{0}F^{20} + L^{0}{}_{3}L^{0}{}_{0}F^{30} + L^{0}{}_{0}L^{0}{}_{1}F^{01} + L^{0}{}_{0}L^{0}{}_{2}F^{02} + $
$+L^{0}{}_{0}L^{0}{}_{3}F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{00} = \frac{-v}{1-v^2}\frac{1}{\sqrt{1-v^2}}(-E_x) + \frac{1}{\sqrt{1-v^2}}\frac{-v}{1-v^2}E_x = 0.$

Now,

$(F')^{32} = L^3{}_1L^2{}_0F^{10} + L^3{}_2L^2{}_0F^{20} + L^3{}_3L^2{}_0F^{30} + L^3{}_0L^2{}_1F^{01} + L^3{}_0L^2{}_2F^{02} + +L^3{}_0L^2{}_3F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{32} = 0.$

The elements of the primed field tensor are almost completed.

$(F')^{21} = L^2{}_1L^1{}_0F^{10} + L^2{}_2L^1{}_0F^{20} + L^2{}_3L^1{}_0F^{30} + L^2{}_0L^1{}_1F^{01} + L^2{}_0L^1{}_2F^{02} + +L^2{}_0L^1{}_3F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{21} = L^2{}_2L^1{}_0F^{20} = \frac{-vE_y}{\sqrt{1-v^2}}.$

Next diagonal element:

$(F')^{11} = L^1{}_1L^1{}_0F^{10} + L^1{}_2L^1{}_0F^{10} + L^1{}_3L^1{}_0F^{30} + L^1{}_0L^1{}_1F^{01} + L^1{}_0L^1{}_2F^{02} + +L^1{}_0L^1{}_3F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{11} = \frac{1}{1-v^2}\frac{-v}{\sqrt{1-v^2}}(-E_x) + \frac{-v}{\sqrt{1-v^2}}\frac{1}{1-v^2}E_x = 0.$

$(F')^{22} = L^2{}_1L^2{}_0F^{10} + L^2{}_2L^2{}_0F^{20} + L^2{}_3L^2{}_0F^{30} + L^2{}_0L^2{}_1F^{01} + L^2{}_0L^2{}_2F^{02} + +L^2{}_0L^2{}_3F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{22} = 0.$

The last element:

$(F')^{33} = L^3{}_1L^3{}_0F^{10} + L^3{}_2L^3{}_0F^{20} + L^3{}_3L^3{}_0F^{30} + L^3{}_0L^3{}_1F^{01} + L^3{}_0L^3{}_2F^{02} + +L^3{}_0L^3{}_3F^{03}.$

Use the elements equal to zero in the Lorentz transformation matrix:

$(F')^{33} = 0.$

All the ingredients for the primed field tensor are available.

$$(F')^{\mu\nu} = \begin{pmatrix} 0 & E_x & \frac{E_y}{\sqrt{1-v^2}} & \frac{E_z}{\sqrt{1-v^2}} \\ -E_x & 0 & \frac{-vE_y}{\sqrt{1-v^2}} & \frac{-vE_z}{\sqrt{1-v^2}} \\ \frac{-E_y}{\sqrt{1-v^2}} & \frac{vE_y}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{-E_z}{\sqrt{1-v^2}} & \frac{vE_z}{\sqrt{1-v^2}} & 0 & 0 \end{pmatrix}.$$

Exercise 8.2 Transformation of a field tensor of the moving frame into the field tensor of a rest frame.

A is sitting in the station as the train passing by. In terms of L's field components, what is the x component of E observed by A? What are the y and z components? What are the components of the magnetic field seen by A?

I assume

- L to be on the moving train,
- L is in the primed position,
- L's field tensor is presented on the lower part of page 275 and primed.

The equation to deal with is:
$F^{\mu\nu} = (L')^{\mu}{}_{\sigma}(L')^{\nu}{}_{\tau}(F')^{\sigma\tau}$,
where $F^{\mu\nu}$ is A's field tensor.
L's field tensor is given:

$$(F')^{\sigma\tau} = \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{pmatrix}.$$

The Lorentz transformation matrix is:

$$(L')^{\mu}{}_{\nu} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

adjusted Eqs.(8.1) and (8.2).
$F^{\mu\nu} = (L')^{\mu}{}_{\sigma}(L')^{\nu}{}_{\tau}(F')^{\sigma\tau}$ consists of 16 terms of which 4 can be deleted on beforehand: the diagonal elements of the field tensor $(F')^{\sigma\tau}$.
Note: I do not use the anti-symmetric nature of the field tensor. Curiosity and a check on the results.
Let's start with the E_x components of A's field tensor, I neglect the primes on the right hand side:

$$F^{01} = L^0{}_1L^1{}_0F^{10} + L^0{}_2L^1{}_0F^{20} + L^0{}_3L^1{}_0F^{30} +$$
$$+L^0{}_0L^1{}_1F^{01} + L^0{}_2L^1{}_1F^{21} + L^0{}_3L^1{}_1F^{31} +$$
$$+L^0{}_0L^1{}_2F^{02} + L^0{}_1L^1{}_2F^{12} + L^0{}_3L^1{}_2F^{32} +$$
$$+L^0{}_0L^1{}_3F^{03} + L^0{}_1L^1{}_3F^{13} + L^0{}_2L^1{}_3F^{23}.$$

In this expression the elements of the Lorentz transformation equal to zero can left out of account:

$$F^{01} = L^0{}_1L^1{}_0F^{10} + L^0{}_0L^1{}_1F^{01}.$$

Now , I plug into this expression the elements of the Lorentz transformation:

$$F^{01} = \frac{v^2}{1-v^2}(-E_x) + \frac{1}{1-v^2}E_x = E_x.$$

Next F^{02}, in A's field tensor:

$$F^{02} = L^0{}_1L^2{}_0F^{10} + L^0{}_2L^2{}_0F^{20} + L^0{}_3L^2{}_0F^{30} +$$
$$+L^0{}_0L^2{}_1F^{01} + L^0{}_2L^2{}_1F^{21} + L^0{}_3L^2{}_1F^{31} +$$
$$+L^0{}_0L^2{}_2F^{02} + L^0{}_1L^2{}_2F^{12} + L^0{}_3L^2{}_2F^{32} +$$
$$+L^0{}_0L^2{}_3F^{03} + L^0{}_1L^2{}_3F^{13} + L^0{}_2L^2{}_3F^{23}.$$

Use the zeroes of the Lorentz transformation:

$$F^{02} = L^0{}_0L^2{}_2F^{02} + L^0{}_1L^2{}_2F^{12} = \frac{E_y}{\sqrt{1-v^2}} + \frac{vB_z}{\sqrt{1-v^2}} = \frac{E_y+vB_z}{\sqrt{1-v^2}}.$$

Then, F^{03}, in A's field tensor:

$$F^{03} = L^0{}_1L^3{}_0F^{10} + L^0{}_2L^3{}_0F^{20} + L^0{}_3L^3{}_0F^{30} +$$
$$+L^0{}_0L^3{}_1F^{01} + L^0{}_2L^3{}_1F^{21} + L^0{}_3L^3{}_1F^{31} +$$
$$+L^0{}_0L^3{}_2F^{02} + L^0{}_1L^3{}_2F^{12} + L^0{}_3L^3{}_2F^{32} +$$
$$+L^0{}_0L^3{}_3F^{03} + L^0{}_1L^3{}_3F^{13} + L^0{}_2L^3{}_3F^{23}.$$

With the zeroes of the Lorentz transformation:

$F^{03} = L^0{}_0 L^3{}_3 F^{03} + L^0{}_1 L^3{}_3 F^{13} = \frac{E_z}{\sqrt{1-v^2}} - \frac{vB_y}{\sqrt{1-v^2}} = \frac{E_z - vB_y}{\sqrt{1-v^2}}$.

Then,

$F^{10} = L^1{}_1 L^0{}_0 F^{10} + L^1{}_2 L^0{}_0 F^{20} + L^1{}_3 L^0{}_0 F^{30} +$
$+ L^1{}_0 L^0{}_1 F^{01} + L^1{}_2 L^0{}_1 F^{21} + L^1{}_3 L^0{}_1 F^{31} +$
$+ L^1{}_0 L^0{}_2 F^{02} + L^1{}_1 L^0{}_2 F^{12} + L^1{}_3 L^0{}_2 F^{32} +$
$+ L^1{}_0 L^0{}_3 F^{03} + L^1{}_1 L^0{}_3 F^{13} + L^1{}_2 L^0{}_3 F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{10} = L^1{}_1 L^0{}_0 F^{10} + L^1{}_0 L^0{}_1 F^{01} = -\frac{1}{1-v^2} E_x + \frac{v^2}{1-v^2} E_x = -E_x.$

Next,

F^{12}, in A's field tensor:

$F^{12} = L^1{}_1 L^2{}_0 F^{10} + L^1{}_2 L^2{}_0 F^{20} + L^1{}_3 L^2{}_0 F^{30} +$
$+ L^1{}_0 L^2{}_1 F^{01} + L^1{}_2 L^2{}_1 F^{21} + L^1{}_3 L^2{}_1 F^{31} +$
$+ L^1{}_0 L^2{}_2 F^{02} + L^1{}_1 L^2{}_2 F^{12} + L^1{}_3 L^2{}_2 F^{32} +$
$+ L^1{}_0 L^2{}_3 F^{03} + L^1{}_1 L^2{}_3 F^{13} + L^1{}_2 L^2{}_3 F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{12} = L^1{}_0 L^2{}_2 F^{02} + L^1{}_1 L^2{}_2 F^{12} = \frac{vE_y}{\sqrt{1-v^2}} + \frac{B_z}{\sqrt{1-v^2}} = \frac{vE_y + B_z}{\sqrt{1-v^2}}$.

Then, F^{13}, in A's field tensor:

$F^{13} = L^1{}_1 L^3{}_0 F^{10} + L^1{}_2 L^3{}_0 F^{20} + L^1{}_3 L^3{}_0 F^{30} +$
$+ L^1{}_0 L^3{}_1 F^{01} + L^1{}_2 L^3{}_1 F^{21} + L^1{}_3 L^3{}_1 F^{31} +$
$+ L^1{}_0 L^3{}_2 F^{02} + L^1{}_1 L^3{}_2 F^{12} + L^1{}_3 L^3{}_2 F^{32} +$
$+ L^1{}_0 L^3{}_3 F^{03} + L^1{}_1 L^3{}_3 F^{13} + L^1{}_2 L^3{}_3 F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{13} = L^1{}_0 L^3{}_3 F^{03} + L^1{}_1 L^3{}_3 F^{13} = \frac{vE_z}{\sqrt{1-v^2}} - \frac{B_y}{\sqrt{1-v^2}} = \frac{vE_z - B_y}{\sqrt{1-v^2}}$.

The next row:

F^{20}, in A's field tensor:

$F^{20} = L^2{}_1 L^0{}_0 F^{10} + L^2{}_2 L^0{}_0 F^{20} + L^2{}_3 L^0{}_0 F^{30} +$
$+ L^2{}_0 L^0{}_1 F^{01} + L^2{}_2 L^0{}_1 F^{21} + L^2{}_3 L^0{}_1 F^{31} +$
$+ L^2{}_0 L^0{}_2 F^{02} + L^2{}_1 L^0{}_2 F^{12} + L^2{}_3 L^0{}_2 F^{32} +$
$+ L^2{}_0 L^0{}_3 F^{03} + L^2{}_1 L^0{}_3 F^{13} + L^2{}_2 L^0{}_3 F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{20} = L^2{}_2 L^0{}_0 F^{20} + L^2{}_2 L^0{}_1 F^{21} = -\frac{E_y}{\sqrt{1-v^2}} - \frac{vB_z}{\sqrt{1-v^2}} = \frac{-E_y - vB_z}{\sqrt{1-v^2}}$.

F^{21}, in A's field tensor:

$F^{21} = L^2{}_1 L^1{}_0 F^{10} + L^2{}_2 L^1{}_0 F^{20} + L^2{}_3 L^1{}_0 F^{30} +$
$+ L^2{}_0 L^1{}_1 F^{01} + L^2{}_2 L^1{}_1 F^{21} + L^2{}_3 L^1{}_1 F^{31} +$
$+ L^2{}_0 L^1{}_2 F^{02} + L^2{}_1 L^1{}_2 F^{12} + L^2{}_3 L^1{}_2 F^{32} +$
$+ L^2{}_0 L^1{}_3 F^{03} + L^2{}_1 L^1{}_3 F^{13} + L^2{}_2 L^1{}_3 F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{21} = L^2{}_2 L^1{}_0 F^{20} + L^2{}_2 L^1{}_1 F^{21} = -\frac{vE_y}{\sqrt{1-v^2}} - \frac{B_z}{\sqrt{1-v^2}} = \frac{-vE_y - B_z}{\sqrt{1-v^2}}$.

Now, F^{23} , in A's field tensor:

$F^{23} = L^2{}_1 L^3{}_0 F^{10} + L^2{}_2 L^3{}_0 F^{20} + L^2{}_3 L^3{}_0 F^{30} +$
$+ L^2{}_0 L^3{}_1 F^{01} + L^2{}_2 L^3{}_1 F^{21} + L^2{}_3 L^3{}_1 F^{31} +$

$+L^2{}_0L^3{}_2F^{02} + L^2{}_1L^3{}_2F^{12} + L^2{}_3L^3{}_2F^{32} +$
$+L^2{}_0L^3{}_3F^{03} + L^2{}_1L^3{}_3F^{13} + L^2{}_2L^3{}_3F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{23} = L^2{}_2L^3{}_3F^{23} = B_x.$

The next row:

F^{30}, in A's field tensor:

$F^{30} = L^3{}_1L^0{}_0F^{10} + L^3{}_2L^0{}_0F^{20} + L^3{}_3L^0{}_0F^{30} +$
$+L^3{}_0L^0{}_1F^{01} + L^3{}_2L^0{}_1F^{21} + L^3{}_3L^0{}_1F^{31} +$
$+L^3{}_0L^0{}_2F^{02} + L^3{}_1L^0{}_2F^{12} + L^3{}_3L^0{}_2F^{32} +$
$+L^3{}_0L^0{}_3F^{03} + L^3{}_1L^0{}_3F^{13} + L^3{}_2L^0{}_3F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{30} = L^3{}_3L^0{}_0F^{30} + L^3{}_3L^0{}_1F^{31} = -\frac{E_z}{\sqrt{1-v^2}} + \frac{vB_y}{\sqrt{1-v^2}} = \frac{-E_z+vB_y}{\sqrt{1-v^2}}.$

Next, F^{31}, in A's field tensor:

$F^{31} = L^3{}_1L^1{}_0F^{10} + L^3{}_2L^1{}_0F^{20} + L^3{}_3L^1{}_0F^{30} +$
$+L^3{}_0L^1{}_1F^{01} + L^3{}_2L^1{}_1F^{21} + L^3{}_3L^1{}_1F^{31} +$
$+L^3{}_0L^1{}_2F^{02} + L^3{}_1L^1{}_2F^{12} + L^3{}_3L^1{}_2F^{32} +$
$+L^3{}_0L^1{}_3F^{03} + L^3{}_1L^1{}_3F^{13} + L^3{}_2L^1{}_3F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{31} = L^3{}_3L^1{}_0F^{30} + L^3{}_3L^1{}_1F^{31} = -\frac{vE_z}{\sqrt{1-v^2}} + \frac{B_y}{\sqrt{1-v^2}} = \frac{-vE_z+B_y}{\sqrt{1-v^2}}.$

Then, F^{32}, in A's field tensor:

$F^{32} = L^3{}_1L^2{}_0F^{10} + L^3{}_2L^2{}_0F^{20} + L^3{}_3L^2{}_0F^{30} +$
$+L^3{}_0L^2{}_1F^{01} + L^3{}_2L^2{}_1F^{21} + L^3{}_3L^2{}_1F^{31} +$
$+L^3{}_0L^2{}_2F^{02} + L^3{}_1L^2{}_2F^{12} + L^3{}_3L^2{}_2F^{32} +$
$+L^3{}_0L^2{}_3F^{03} + L^3{}_1L^2{}_3F^{13} + L^3{}_2L^2{}_3F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{32} = L^3{}_3L^2{}_2F^{32} = -B_x.$

The ingredients for $F^{\mu\nu}$ are available.

Are they? Well, Let's look for F^{00}.

$F^{00} = L^0{}_1L^0{}_0F^{10} + L^0{}_2L^0{}_0F^{20} + L^0{}_3L^0{}_0F^{30} +$
$+L^0{}_0L^0{}_1F^{01} + L^0{}_2L^0{}_1F^{21} + L^0{}_3L^0{}_1F^{31} +$
$+L^0{}_0L^0{}_2F^{02} + L^0{}_1L^0{}_2F^{12} + L^0{}_3L^0{}_2F^{32} +$
$+L^0{}_0L^0{}_3F^{03} + L^0{}_1L^0{}_3F^{13} + L^0{}_2L^0{}_3F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{00} = L^0{}_1L^0{}_0F^{10} + L^0{}_0L^0{}_1F^{01} = -\frac{vE_x}{1-v^2} + \frac{vE_x}{1-v^2} = 0.$

Next,

$F^{11} = L^1{}_1L^1{}_0F^{10} + L^1{}_2L^1{}_0F^{20} + L^1{}_3L^1{}_0F^{30} +$
$+L^1{}_0L^1{}_1F^{01} + L^1{}_2L^1{}_1F^{21} + L^1{}_3L^1{}_1F^{31} +$
$+L^1{}_0L^1{}_2F^{02} + L^1{}_1L^1{}_2F^{12} + L^1{}_3L^1{}_2F^{32} +$
$+L^1{}_0L^1{}_3F^{03} + L^1{}_1L^1{}_3F^{13} + L^1{}_2L^1{}_3F^{23}.$

With the zeroes of the Lorentz transformation:

$F^{11} = L^1{}_1L^1{}_0F^{10} + L^1{}_0L^1{}_1F^{01} = -\frac{vE_x}{1-v^2} + \frac{vE_x}{1-v^2} = 0 .$

Without further details:

$F^{22} = F^{33} = 0.$

A's Field tensor:

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x & \frac{E_y+vB_z}{\sqrt{1-v^2}} & \frac{E_z-vB_y}{\sqrt{1-v^2}} \\ -E_x & 0 & \frac{vE_y+B_z}{\sqrt{1-v^2}} & \frac{vE_z-B_y}{\sqrt{1-v^2}} \\ \frac{-E_y-vB_z}{\sqrt{1-v^2}} & \frac{-vE_y-B_z}{\sqrt{1-v^2}} & 0 & B_x \\ \frac{-E_z+vB_y}{\sqrt{1-v^2}} & \frac{-vE_z+B_y}{\sqrt{1-v^2}} & -B_x & 0 \end{pmatrix}.$$

8.1.2 Summary of Einstein's Example

"The force on the(moving) electron is due to a magnetic field in the laboratory frame is due to an electric field in the moving frame (the electron's rest frame)."

Exercise 8.3: About Einstein's example, Thought Experiment.

In lecture 8.1, Einstein's thought experiment is described. In the laboratory frame the magnet is at rest. The constant magnetic field has one component: B_z. The electron is moving with velocity v in the positive x-direction, Figure 8.1. In the frame of the electron, the primed frame, the case is illustrated in Figure 8.2. The electron is at rest and the magnet moves to the left with velocity v .

Now, for Einstein's example, thought experiment, work out all components and magnetic fields in the electron's rest frame, the primed frame.

Note: In this exercise I will use the anti-symmetric nature of the field tensor.

The unprimed field tensor, in the laboratory frame:

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +B_z & 0 \\ 0 & -B_z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Lorentz transformation tensor:

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2}} & \frac{-v}{\sqrt{1-v^2}} & 0 & 0 \\ \frac{-v}{\sqrt{1-v^2}} & \frac{1}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The equation to deal with :

$(F')^{\mu\nu} = L^{\mu}{}_{\sigma} L^{\nu}{}_{\tau} F^{\sigma\tau}$, $\sigma, \tau = 0,1,2,3.$

Only F^{21} and F^{12} do contribute.

The first row:

$(F')^{01} = L^{0}{}_{2} L^{1}{}_{1} F^{21} + L^{0}{}_{1} L^{1}{}_{2} F^{12}.$

Then,

$(F')^{01} = 0.$

$(F')^{02} = L^{0}{}_{2} L^{2}{}_{1} F^{21} + L^{0}{}_{1} L^{2}{}_{2} F^{12}.$

Then,

$(F')^{02} = \frac{-vB_z}{\sqrt{1-v^2}}$, Eq.(8.4).

$(F')^{03} = L^0{}_2 L^3{}_1 F^{21} + L^0{}_1 L^3{}_2 F^{12}$.
So,
$(F')^{03} = 0$.

The second row:
$(F')^{12} = L^1{}_2 L^2{}_1 F^{21} + L^1{}_1 L^2{}_2 F^{12} = \frac{B_z}{\sqrt{1-v^2}}$.

$(F')^{13} = L^1{}_2 L^3{}_1 F^{21} + L^1{}_1 L^3{}_2 F^{12} = 0$.

The third row:
$(F')^{23} = L^2{}_2 L^3{}_1 F^{21} + L^2{}_1 L^3{}_2 F^{12} = 0$.

The fourth row:
No contribution.

The field tensor in the electron's rest frame:

$$(F')^{\mu\nu} = \begin{pmatrix} 0 & 0 & \frac{-vB_z}{\sqrt{1-v^2}} & 0 \\ 0 & 0 & \frac{B_z}{\sqrt{1-v^2}} & 0 \\ \frac{vB_z}{\sqrt{1-v^2}} & \frac{-B_z}{\sqrt{1-v^2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

8.2 Introduction to Maxwell's Equations

"…. charges control fields through Maxwell's equations".

8.2.1 Vector Identities

"An identity is a mathematical fact that follows from a definition".

Two identities.

The First: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$, *Proof by brute force.*

The Second: $\vec{\nabla} \times (\vec{\nabla} S) = 0$.

8.2.2 Magnetic Field

We have:

$\vec{\nabla} \cdot \vec{B} = 0$, Eq. (8.5).

$\Longrightarrow$ "*There cannot be magnetic charges*".

8.2.3 Electric Field

The definition of the electric field is used:

$E_n = -\left(\frac{\partial A_n}{\partial t} - \frac{\partial A_0}{\partial X^n}\right)$, Eq. (8.6).

8.2.4 Two More Maxwell Equations

In table 8.1, page 285, the Maxwell Equations are summarized.

8.2.5 Charge Density and Current Density

Charge density ρ is the limiting value of the charge ΔQ in the volume element divided by this volume element ΔV.

Current density $\vec{j}$; the amount of charge ΔQ flowing through a window ΔA per unit time Δt. Note: A is here the notation for area.

8.2.6 Conservation of Charge

In formula:

$\frac{dQ}{dt} = 0$, Eq.(8.16).

Conservation of charge means: local conservation of charge.

On basis of Figure 8.6, the model for *Local Charge Conservation,* Susskind derived the differential equation for this conservation, Eq.(8.23).

Some remarks:

Page 294, "*How much charge flows into the box from the right?"*

The expression

$\Delta Q_{right} = -(j_{x-})\Delta y \Delta z \Delta t \Longrightarrow -(j_{x+})\Delta y \Delta z \Delta t$,

"*…, the amount entering from the left is*"

$\Delta Q_{right} = (j_{x+})\Delta y \Delta z \Delta t \Longrightarrow (j_{x-})\Delta y \Delta z \Delta t.$

See Figure 8.6.

Then, the expression at the top of page 295 becomes:

$-(j_{x+} - j_{x-}) = -\frac{\partial j_x}{\partial x}\Delta x.$

Resulting into Eq.(8.18).

So, does it matter? No, but I prefer the above notation at the right-hand side of the arrow.

Exercise 8.4 The continuity equation or the equation of local conservation

Use the second group of Maxwell's equations from Table 8.1 along with two vector identities from section 8.2.1 to derive the continuity equation (local conservation).

I use: *"…. the divergence of a curl is always zero*", page 280.

So,

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$.

Then, with

$\vec{\nabla} \times \vec{B} - \frac{\partial}{\partial t}\vec{E} = \vec{j}$,

from Table 8.1,

use the divergence of a curl is zero:

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) - \vec{\nabla} \cdot \frac{\partial}{\partial t}\vec{E} = \vec{\nabla} \cdot \vec{j} \Longrightarrow \frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{E} + \vec{\nabla} \cdot \vec{j} = 0$.

Furthermore,

$\vec{\nabla} \cdot \vec{E} = \rho$,

from Table 8.1.

Plug this expression into the preceding equation:

$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \Longrightarrow$ Eq.(8.23).

8.2.7 Maxwell's Equations: Tensor Form

The first group of Maxwell' equation and the continuity equation are written in tensor form.

The first group, Table 8.1<

$\vec{\nabla} \cdot \vec{B} = 0,$

$\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0,$
and
$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$.
The new object J^{μ} is defined resulting into Eq.(.24)
$\frac{\partial J^{\mu}}{\partial X^{\mu}} = 0$, Eq.(8.24).
Then, Susskind proved the components of J^{μ} transform as the components of a 4-vector.
In Eq. (8.25) the components of the 4-vector J^{μ} are presented.

8.2.8 The Bianchi Identity.

Eq.(6.41):

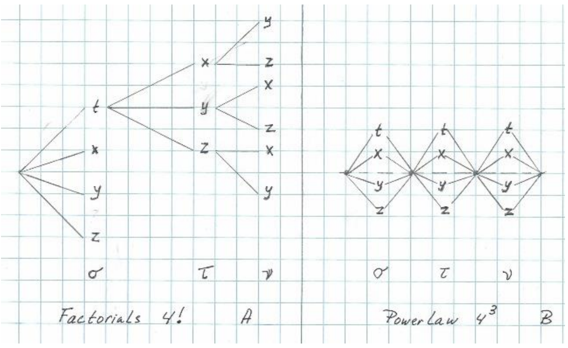
$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ +E_x & 0 & +B_z & -B_y \\ +E_y & -B_z & 0 & +B_x \\ +E_z & +B_y & -B_x & 0 \end{pmatrix}.$$

Page 300, at the bottom,
$F_{yz} = F_{23} = B_x$,
$F_{zx} = F_{31} = B_y$,
$F_{xy} = F_{12} = B_z$.
With these expressions the divergence of $\vec{B}$ is obtained equivalent to Eq. (8.26).
Now, set
$\sigma = z$,
$\nu = y$,
$\tau = x$.
With Eq.(8.28):
$\partial_z F_{yx} + \partial_y F_{xz} + \partial_x F_{zy} = 0$.
$F_{yx} = F_{21} = -B_z$,
$F_{xz} = F_{13} = -B_y$,
$F_{zy} = F_{32} = -B_x$.
Then, $\partial_z F_{yx} + \partial_y F_{xz} + \partial_x F_{zy} = 0 \Longrightarrow -\partial_z B_z - \partial_y B_y - \partial_x B_x = 0 \Longrightarrow$
$\Longrightarrow \partial_x B_x + \partial_y B_y + \partial_z B_z = 0.$
Next the time component:
$\sigma = y = 2$,
$\nu = x = 1$,
$\tau = t = 0$.
With Eq.(6.41):
$F_{\nu\tau} = F_{10} = E_x$,
$F_{\tau\sigma} = F_{03} = -E_y$,
$F_{\sigma\nu} = F_{12} = -B_z$, page 301.
$\partial_\sigma = \partial_y$, $\partial_\nu = \partial_x$ and $\partial_\tau = \partial_t$.
Plugging these results in Eq.(8.28) the z component of $\vec{\nabla} \times \vec{E} + \frac{\partial}{\partial t}\vec{B} = 0$:
$(\vec{\nabla} \times \vec{E})_z + \partial_t B_z = 0$,
appears.

Top of page 302:
"How many ways are there to assign values to σ, ν and τ? There are three indices, and each index can be assigned one of the four values t, x, y and z. That means there are $4 \times 4 \times 4 = 64$, different ways to do it".
A closer look.

Intermezzo Permutations

There are three indices, and each index can be assigned one of the four values t, x, y and z.
The permutations are shown in the Figure below.



In A of the Figure there are 24(4!) possibilities shown. In B of the same figure 64(4^3) possibilities.
On the left-hand side we have, e.g.,
$\sigma = t$,
$\nu = y$,
$\tau = x$.
On the right-hand side we have, e.g.,
$\sigma = t$,
$\nu = t$,
$\tau = t$.
So, I think the factorial are meant, but I am not so sure.
End of Intermezzo.

On page 302, below the middle of the page, Susskind introduced another way to check the Bianchi identity. The definition of $F_{\mu\nu}$, Eqs. (7.6) and (7.9), is used. This expression for $F_{\mu\nu}$ is plugged into Eq. (8.28). This results into the equation at the bottom of page 302.
Use:
$\sigma = z$,
$\nu = y$,
$\tau = x$,
in the equation at the bottom of page 302 and all the terms cancel. As mentioned by Susskind, the relevant equation can be expanded, meaning,
$\partial_\sigma = \frac{\partial}{\partial X^\sigma}$, etc, again all the terms cancel.

Lecture 9 Physical Consequences of Maxwell's Equations

9.1 Mathematical Interlude

The fundamental theorem of calculus is presented, Eq. (9.1).

9.1.1 Gauss's Theorem

In formula, Eq. (9.2):

$\int \vec{\nabla} \cdot \vec{V} d^3x = \int \vec{V} \cdot \hat{n} dS$,

where,

$\vec{V}$ is a vector field,

S is a surface of the volume,

and

$\hat{n}$ is an outward pointing unit normal vector on the surface.

On page 306, Susskind presented the spherical symmetric case.

Then, Eq. (9.3) represents Gauss's theorem of a spherical symmetrical vector field.

9.1.2 Stokes's Theorem

In formula, Eq. (9.4):

$\int (\vec{\nabla} \times \vec{V}) \cdot \hat{n} dS = \oint \vec{V} \cdot \overrightarrow{dl}$, or $\widehat{dl}$

where $\widehat{dl}$ is the unit vector along the bounding curve of the surface S.

9.1.3 Theorem Without a Name

Eq. (9.7), the unnamed theorem:

$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla}^2 \vec{V}$.

The proof of this theorem is by brute force: write out all the terms explicitly and compare both sides.

Proof of theorems

On page 280, Susskind wrote to prove

$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$,

use brute force.

Do we need to apply brute force?

Let us formulate $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$, a bit more general and use the vectors $\vec{u}$ and $\vec{v}$.

So,

$\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

We know de vector $\vec{u} \times \vec{v}$ to be ⊥ to $\vec{u}$ and $\vec{v}$. Consequently, $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$.

Now, the other theorem without Name:

$\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \vec{\nabla}^2 \vec{V}$.

I use again the vectors $\vec{u}$ and $\vec{v}$. May be a bit more brute force.

$\vec{u} \times (\vec{u} \times \vec{v}) = \vec{u}(\vec{u} \cdot \vec{v}) - \vec{u}^2 \vec{v}$.

In elements of the vectors on the left hand side:

$\vec{u} \times (\vec{u} \times \vec{v}) = (u_1, u_2, u_3) \times (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1) =$
$= [u_2(u_1v_2 - u_2v_1) - u_3(u_3v_1 - u_1v_3), u_3(u_2v_3 - u_3v_2) - u_1(u_1v_2 - u_2v_1),$
$u_1(u_3v_1 - u_1v_3) - u_2(u_2v_3 - u_3v_2)]$.

The right hand side:

$\vec{u}(\vec{u} \cdot \vec{v}) - \vec{u}^2 \vec{v} = [u_1(u_1v_1 + u_2v_2 + u_3v_3) - (u_1u_1 + u_2u_2 + u_3u_3)v_1,$
$u_2(u_1v_1 + u_2v_2 + u_3v_3) - (u_1u_1 + u_2u_2 + u_3u_3)v_2,$
$u_3(u_1v_1 + u_2v_2 + u_3v_3) - (u_1u_1 + u_2u_2 + u_3u_3)v_3] =$
$= [u_1(u_2v_2 + u_3v_3) - (u_2u_2 + u_3u_3)v_1, u_2(u_1v_1 + u_3v_3) - (u_1u_1 + u_3u_3)v_2,$

$u_3(u_1v_1 + u_2v_2) - (u_1u_1 + u_2u_2)v_3] = [u_2(u_1v_2 - u_2v_1) - u_3(u_3v_1 - u_1v_3),$
$u_3(u_2v_3 - u_3v_2) - u_1(u_1v_2 - u_2v_1), u_1(u_3v_1 - u_1v_3) - u_2(u_2v_3 - u_3v_2)].$
Hence,
$\vec{u} \times (\vec{u} \times \vec{v}) = \vec{u}(\vec{u} \cdot \vec{v}) - \vec{u}^2\vec{v}$.
End of Proof.

9.2 Laws of Electrodynamics

9.2.1 The Conservation of Electric Charge

The basic equation is the continuity equation, Eq. (8.23),
$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0$.
Susskind integrated this expression over a volume bounded by a surface S. Gauss's theorem is applied, and Eq. (9.9) is obtained.
"The change of charge in a volume is equal the flow of charge through the surface of the volume".

9.2.2 From Maxwell's Equations to the laws of Electrodynamics

The equations given in Table 8.1, page 285, are summarized included with factors of c , Eq. (9.10).

9.2.3 Coulomb's Law.

Coulomb's law is derived from first principles.
A three dimensional δ-function is used for a point charge Q, Eq. (9.11).
Then, Maxwell's third equation is used:
$\vec{\nabla} \cdot \vec{E} = \rho$, Eq. (9.10).
The electric field vector is plugged into Eq.(9.3): $\vec{V} \Longrightarrow \vec{E}$.
The volume integral of the charge density is taken using the 3—dimensional δ-function. This produces Eq. (9.12).
With a second charge at a distance r from Q, the force law is found, Eq.(9.13).

9.24 Faraday's Law

The electromotive force is introduced: the integral around a closed loop, Eq. (9.15).
With Stokes theorem and Maxwell's equations the equation for the EMF is presented in a concise form in Eq. (9.18), using the magnetic flux
$\Phi = -\frac{d}{dt}\int \vec{B} \cdot \hat{n} dS$.

9.25 Ampère's Law

Maxwell's fourth equation without time dependency is:
$\vec{\nabla} \times \vec{B} = \frac{\vec{j}}{c^2}$, Eq. (9.19).
Using Stokes theorem, Eq. (9.21).
In words: *It follows that a current through a "wire" produces a magnetic field that circulate around the "wire". The "wire" is an imaginary mathematical circle*.

9.26 Maxwell's Law

In this section Susskind shows Maxwell's Equations imply that the electric and magnetic fields satisfy the wave equation.

Maxwell's equations are analysed for $\rho = 0$, and $\vec{j} = 0$, Eq. (9.10) $\Longrightarrow$ (9.23).
Then, almost magically, Susskind obtains the wave equation, Eq. (9.25).

Lecture 10 Maxwell From Lagrange

In the table on page 326, the set of Maxwell equation is summarized.

The 4-vector notation of $\vec{\nabla} \times \vec{B} - \frac{\partial}{\partial t}\vec{E} = \vec{j}$, is presented for the first time:

$\partial_\nu F^{\mu\nu} = J^\mu$.

10.1 Electromagnetic waves

Susskind started with the four Maxwell equations without the source term.
A generic plane wave is used. The resulting components for the electric field are given in Eqs. (10.1)-(10.3).

The polarisation vector $\vec{\mathcal{E}}$, lies along the x-axis. The, the form of the electric field is given by Eq. (10.5).
The magnetic field is described by the Eqs. (10.6)-(10.8).
Furthermore, the magnetic field must be perpendicular to the electric field. The relevant component of the Maxwell equation is:

$\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}$, Eq. (10.9) .
With

$E_x = \mathcal{E}_x \sin(kz - \omega t)$,
andω

$B_y = \mathcal{B}_y \sin(kz - \omega t)$,

$\frac{\partial B_y}{\partial t} = \frac{\partial E_x}{\partial z} \Longrightarrow \mathcal{B}_y = -\frac{k}{\omega}\mathcal{E}_x$, Eq.(10.10).
Notice at the top of page 330 a few typo's .
Then one more Maxwell equation is analysed:

$c^2 \cdot \vec{\nabla} \times \vec{E} - \frac{\partial}{\partial t}\vec{B} = 0$.
From the x-component, we find:

$\frac{\partial E_x}{\partial t} = c^2 \cdot (\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z})$.

Susskind writes: "*.. that it reduces to* $\frac{\partial E_x}{\partial t} = -\frac{\partial B_y}{\partial t}$."

Plug into $\frac{\partial E_x}{\partial t} = -\frac{\partial B_y}{\partial t}$, $E_x = \mathcal{E}_x \sin(kz - \omega t)$ and $B_y = \mathcal{B}_y \sin(kz - \omega t) \Longrightarrow\ \mathcal{E}_x = -\mathcal{B}_y$.

This contradicts Eq. (10.10), unless $\frac{k}{\omega} = 1$.

There is more, with $\frac{\partial E_x}{\partial t} = -c^2 \cdot \frac{\partial B_y}{\partial z} \Longrightarrow -\omega\mathcal{E}_x \cos(kz - \omega t) = -\, c^2 k \mathcal{B}_y \cos(kz - \omega t) \Longrightarrow$
$\Longrightarrow\ \omega\mathcal{E}_x = c^2 k \mathcal{B}_y$.
With Eq. (10.10) , the preceding expression results into:

$\omega\mathcal{E}_x = -c^2 k \frac{k}{\omega}\mathcal{E}_x \Longrightarrow \omega^2 = -c^2 k^2$.
The minus sign in the preceding expression is not nice.
to deal with that, I used a trick with eq. (10.10):

$\mathcal{B}_y = -\frac{k}{\omega}\mathcal{E}_x \Longrightarrow {\mathcal{B}_y}^2 = (\frac{k}{\omega})^2 {\mathcal{E}_x}^2$.
The same trick with:

$\omega\mathcal{E}_x = c^2 k\mathcal{B}_y \Longrightarrow (\omega\mathcal{E}_x)^2 = (c^2k)^2{\mathcal{B}_y}^2$.

Then, plug ${\mathcal{B}_y}^2 = (\frac{k}{\omega})^2{\mathcal{E}_x}^2$, into $(\omega\mathcal{E}_x)^2 = (c^2k)^2{\mathcal{B}_y}^2 \Longrightarrow c^4 = \left(\frac{\omega}{k}\right)^4 \Longrightarrow c = \frac{\omega}{k}$.

The picture in Figure 10.1 is also illustrated in the video on Maxwell's equations
https://en.wikipedia.org/wiki/Maxwell%27s_equations

10.2 Lagrangian Formulation of Electrodynamics

It is again about the principle of least action. So, we need the Lagrangian.
The action principle is needed to derive Eqs. (10.12) and (10.13).

$\vec{\nabla}\cdot\vec{E} = \rho$,

and

$\vec{\nabla}\times\vec{B} - \frac{\partial}{\partial t}\vec{E} = \vec{j}$.

Eq. (10.14):

"*All these relations captive in covariant form by a single equation*":

$\partial_\mu F^{\mu\nu} = -J^\nu$,

where, Eq. (6.42),

$$F^{\mu\nu} = \begin{pmatrix} 0 & +E_x & +E_y & +E_z \\ -E_x & 0 & +B_z & -B_y \\ -E_y & -B_z & 0 & +B_x \\ -E_z & +B_y & -B_x & 0 \end{pmatrix}$$

Let us look into some elements of Eq. (10.14), with Eq. (6.42):

$\nu = 0:\ \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = -J^0 \Longrightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho \Longrightarrow$

$\Longrightarrow \vec{\nabla}\cdot\vec{E} = \rho$.

Next,

$\nu = 1:\ \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} = -J^1 \Longrightarrow \frac{\partial E_x}{\partial t} - \frac{\partial B_z}{\partial y} + \frac{\partial B_y}{\partial z} = -J^1 \Longrightarrow$

$\Longrightarrow -\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = J^1 \Longrightarrow$, etc,

$\Longrightarrow \vec{\nabla}\times\vec{B} - \frac{\partial}{\partial t}\vec{E} = \vec{j}$.

Now, the action principle is applied.

10.2.1 Locality

We need to get used to the notation:

$\frac{\partial\phi}{\partial X^\mu} = \partial_\mu\phi = \phi_{,\mu}$.

Note: page 334, the middle, $\frac{\partial\phi}{X^\mu} = \partial_\mu\phi = \phi_{,\mu}$. A typo, I suppose.

The action integral:

$Action = \int d^4x\mathcal{L}(\phi,\phi_{,\mu})$.

10.2.2 Lorentz Invariance

$\Longrightarrow$ *the Lagrangian density needs to be a scalar*.

At the bottom of page 335, Susskind presented an expression which could appear in the Lagrangian. We met something like this expression in Lecture 4, Eq. (4.7), section 4.3.3.

In Eq. (10.15), Eq. (4.7) is presented in the new notation:
$\mathcal{L} = -\frac{1}{2}\phi_{,\mu}\phi^{,\mu} - U(\phi)$.
The next step is, working with the new notation, to develop the Euler-Lagrange equation : Eq. (10.16).
On page 337, Susskind presented the Euler-Lagrange equation as obtained in section 4.3.3.

10.2.3 Gauge Invariance

The obvious choices are the components of $F_{\mu\nu}$.
"Therefore, any Lagrangian we construct from the components of F will be gauge invariant".

10.2.4 The Lagrangian in the Absence of Sources

Absence of sources means the current 4-vector is zero:
$J^{\mu} = 0$.
How to construct a meaningful scalar with the field tensor? Page 339: *"… any terms linear in $F_{\mu\nu}$ would not be a good choice…".*
Then, a combination of $F_{\mu\nu}$ is selected: $F_{\mu\nu}F^{\mu\nu}$.
Let us find out what will be the result with Eqs. (6.41) and (6.42),
$F_{\mu\nu}F^{\mu\nu}$:
$\mu = 0\ , F_{00}F^{00} + F_{01}F^{01} + F_{02}F^{02} + F_{03}F^{03} +$
$\mu = 1\ , F_{10}F^{10} + F_{11}F^{11} + F_{12}F^{12} + F_{13}F^{13} +$
$\mu = 2\ , F_{20}F^{20} + F_{21}F^{21} + F_{22}F^{22} + F_{23}F^{23} +$
$\mu = 3\ , F_{30}F^{30} + F_{31}F^{31} + F_{32}F^{32} + F_{33}F^{33} =$
$= 0 - E_x^2 - E_y^2 - E_z^2 +$
$-E_x^2 + 0 + B_z^2 + B_y^2 +$
$-E_y^2 + B_z^2 + 0 + B_x^2 +$
$-E_z^2 + B_y^2 + B_x^2 + 0 = -2\left(E_x^2 + E_y^2 + E_z^2\right) + 2\left(B_x^2 + B_y^2 + B_z^2\right) = -2E^2 + 2B^2.$
This, $-2E^2 + 2B^2$, results into Eq. (10.18), where use has been made of the convention:
$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \Longrightarrow \mathcal{L} = \frac{1}{2}(E^2 - B^2).$

10.3 Deriving Maxwell's Equations

Susskind starts with $J^{\mu} = 0$.
The Euler-Lagrange equations for fields will be derived:
$\frac{\partial}{\partial X^{\nu}}\frac{\partial \mathcal{L}}{\partial \phi_{,\nu}} = \frac{\partial \mathcal{L}}{\partial \phi}$, Eq.(10.19).
These fields are the vector potential components.
Then, for convenience the comma notation for derivatives is extended:
$A_{\mu,\nu} \equiv \frac{\partial A_{\mu}}{\partial X^{\nu}}$.
With this new notation, the field tensor $F_{\mu\nu}$ is presented in Eq. (10.20).
The Lagrangian with this notation is given in Eq. (10.21).
To illustrate the resulting Euler-Lagrange equations, Susskind showed how to arrive there by starting with:
$\frac{\partial \mathcal{L}}{\partial A_{x,\mu}}$.
At the top of page 344, Susskind explained: "*We could write out all sixteen terms of the*

expansion and then look for the terms that contain $A_{x,y}$……”. Then Susskind explained this not to be necessary.
Well, I did write out all the terms of Eq. (10.21):

$$\mathcal{L} = -\frac{1}{4}\left(A_{\nu,\mu} - A_{\mu,\nu}\right)\left(A^{\nu,\mu} - A^{\mu,\nu}\right) = \frac{1}{4}\left(-A_{\nu,\mu}A^{\nu,\mu} + A_{\mu,\nu}A^{\nu,\mu} + A_{\nu,\mu}A^{\mu,\nu} - A_{\mu,\nu}A^{\mu,\nu}\right),$$

with $\mu, \nu \Longrightarrow 0,1,2,3$.

Intermezzo Elements of the Lagrangian

Let’s explore $+\frac{1}{4}A_{\nu,\mu}A^{\mu,\nu}$:
Table 1.

	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
$\nu = 0$	$A_{0,0}A^{0,0} +$	$A_{0,1}A^{1,0} +$	$A_{0,2}A^{2,0} +$	$A_{0,3}A^{3,0} +$
$\nu = 1$	$A_{1,0}A^{0,1} +$	$A_{1,1}A^{1,1} +$	$\mathbf{A_{1,2}A^{2,1}} +$	$A_{1,3}A^{3,1} +$
$\nu = 2$	$A_{2,0}A^{0,2} +$	$\mathbf{A_{2,1}A^{1,2}} +$	$A_{2,2}A^{2,2} +$	$A_{2,3}A^{3,2} +$
$\nu = 3$	$A_{3,0}A^{0,3} +$	$A_{3,1}A^{1,3} +$	$A_{3,2}A^{2,3} +$	$A_{3,3}A^{3,3}$

There are three other tables for: $-\frac{1}{4}A_{\nu,\mu}A^{\nu,\mu}$, $\frac{1}{4}A_{\mu,\nu}A^{\nu,\mu}$, and $-\frac{1}{4}A_{\mu,\nu}A^{\nu,\mu}$.
So, let’s investigate $+\frac{1}{4}A_{\mu,\nu}A^{\nu,\mu}$:
Table 2.

	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
$\nu = 0$	$A_{0,0}A^{0,0} +$	$A_{1,0}A^{0,1} +$	$A_{2,0}A^{0,2} +$	$A_{3,0}A^{0,3} +$
$\nu = 1$	$A_{0,1}A^{1,0} +$	$A_{1,1}A^{1,1} +$	$\mathbf{A_{2,1}A^{1,2}} +$	$A_{3,1}A^{1,3} +$
$\nu = 2$	$A_{0,2}A^{2,0} +$	$\mathbf{A_{1,2}A^{2,1}} +$	$A_{2,2}A^{2,2} +$	$A_{3,2}A^{2,3} +$
$\nu = 3$	$A_{0,3}A^{3,0} +$	$A_{1,3}A^{3,1} +$	$A_{2,3}A^{3,2} +$	$A_{3,3}A^{3,3} +$

Next, $-\frac{1}{4}A_{\nu,\mu}A^{\nu,\mu}$:
Table 3

	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
$\nu = 0$	$A_{0,0}A^{0,0} +$	$A_{0,1}A^{0,1} +$	$A_{0,2}A^{0,2} +$	$A_{0,3}A^{0,3} +$
$\nu = 1$	$A_{1,1}A^{1,0} +$	$A_{1,1}A^{1,1} +$	$\mathbf{A_{1,2}A^{1,2}} +$	$A_{1,3}A^{1,3} +$
$\nu = 2$	$A_{2,2}A^{2,0} +$	$\mathbf{A_{2,1}A^{2,1}} +$	$A_{2,2}A^{2,2} +$	$A_{2,3}A^{2,3} +$
$\nu = 3$	$A_{3,3}A^{3,0}$	$A_{3,1}A^{3,1}$	$A_{3,2}A^{3,2}$	$A_{3,3}A^{3,3}$

Finally, $-\frac{1}{4}A_{\mu,\nu}A^{\nu,\mu}$:
Table 4

	$\mu = 0$	$\mu = 1$	$\mu = 2$	$\mu = 3$
$\nu = 0$	$A_{0,0}A^{0,0} +$	$A_{1,0}A^{0,1} +$	$A_{2,0}A^{0,2} +$	$A_{3,0}A^{0,3} +$
$\nu = 1$	$A_{0,1}A^{1,0} +$	$A_{1,1}A^{1,1} +$	$\mathbf{A_{2,1}A^{1,2}} +$	$A_{3,1}A^{1,3} +$
$\nu = 2$	$A_{0,2}A^{2,0} +$	$\mathbf{A_{1,2}A^{2,1}} +$	$A_{2,2}A^{2,2} +$	$A_{3,2}A^{2,3} +$
$\nu = 3$	$A_{0,3}A^{3,0} +$	$A_{1,3}A^{3,1} +$	$A_{2,3}A^{3,2} +$	$A_{3,3}A^{3,3} +$

End of Intermezzo

In total 64 terms in the above four tables.
Basically, it is about:
for $\mu, \nu \Longrightarrow \ \alpha, \beta$:
$\frac{1}{4}\left(-A_{\beta,\alpha}A^{\beta,\alpha} + A_{\alpha,\beta}A^{\beta,\alpha} + A_{\beta,\alpha}A^{\alpha,\beta} - A_{\alpha,\beta}A^{\alpha,\beta}\right)$.
For $\mu, \nu \Longrightarrow \ \beta, \alpha$:
$\frac{1}{4}\left(-A_{\alpha,\beta}A^{\alpha,\beta} + A_{\beta,\alpha}A^{\alpha,\beta} + A_{\alpha,\beta}A^{\beta,\alpha} - A_{\beta,\alpha}A^{\beta,\alpha}\right)$.
Both preceding expressions combined:
$\frac{1}{4}\left(-2A_{\beta,\alpha}A^{\beta,\alpha} + 2A_{\alpha,\beta}A^{\beta,\alpha} + 2A_{\beta,\alpha}A^{\alpha,\beta} - 2A_{\alpha,\beta}A^{\alpha,\beta}\right)$.
So, the result:
$\frac{1}{2}\left(-A_{\beta,\alpha}A^{\beta,\alpha} + A_{\alpha,\beta}A^{\beta,\alpha} + A_{\beta,\alpha}A^{\alpha,\beta} - A_{\alpha,\beta}A^{\alpha,\beta}\right)$.
This is the expression to be analysed for one combination of $\mu, \nu \Longrightarrow \ \alpha, \beta$.
For $\mu = \nu$,
$\mathcal{L} = 0$.
For $A_{1,2}$, $A_{2,1}$, $A^{1,2}$ and $A^{2,1}$, $x = 1, y = 2$, we have:
$\frac{1}{2}\left(-A_{1,2}A^{1,2} + A_{2,1}A^{1,2} + A_{1,2}A^{2,1} - A_{2,1}A^{2,1}\right) = -\frac{1}{2}\left(A_{1,2} - A_{2,1}\right)(A^{1,2} - A^{2,1})$,
where $A_{1,2}A^{2,1}$, and $A_{2,1}A^{1,2}$, are given in the table above.

Next, Susskind lowered the upper indices, since these are space indices.
Question:
Not all the elements of the field tensor are represented by space indices. What to do about them? What to do with the mixed indices 0,1 or 1,0 , for that matter?
On page 345, the results for space indices are generalized for all the indices μ, ν.
Then, finally the Euler-Lagrange equations for $J^{\mu} = 0$, are obtained.
So, there remain a few questions in the derivation of the Euler -Lagrange equation.
Can these be solved?
In the middle of page 345:
"…or, using antisymmetry of F, …… $F_{xy} = F^{xy}$." I suppose this reflects the change
$-F_{yx} = F_{xy}$, and $-F^{yx} = F^{xy}$.
Obviously, for space indices there is no difference between subscript and superscript, Eqs. (6.41) and (6.42): $F_{xy} = F^{xy} = +B_z$.
Page 227:
"Antisymmetric tensors have the property
$F^{\mu\nu} = -F^{\nu\mu}$*."*
Then, antisymmetry gives back:
$F^{xy} = -F^{yx} = +B_z$,
with no difference between subscript and superscript for space indices, we have:
$F_{xy} = F^{xy} = -F^{yx} = +B_z$.
This leaves the problem of mixed indices unsolved. The problem is not understanding the step made to obtain Eq. (10.22).
Let us analyse field components with mixed time and space components:

$-\frac{1}{2}(A_{0,1} - A_{1,0})(A^{0,1} - A^{1,0})$.
Then, we need, e.g.,
$\frac{\partial \mathcal{L}}{\partial A_{0,1}}$, or $\frac{\partial \mathcal{L}}{\partial A_{1,0}}$.
In the analysis on page 344, Susskind changed $A^{x,y}$, into $A_{x,y}$. Since it is about space indices.
At the bottom of page 344, it is shown that the differentiation of $\mathcal{L}$ with respect to $A_{x,y}$, is straight forward.
How to handle differentiation of $\mathcal{L}$ with respect to $A_{0,1}$ or $A_{1,0}$, for mixed coordinates?
I assume:
$A^{0,1} = -A_{0,1}$, and $A^{1,0} = -A_{1,0}$,
related with lowering the time index.
With these results:
$-\frac{1}{2}(A_{0,1} - A_{1,0})(A^{0,1} - A^{1,0}) \Longrightarrow \frac{1}{2}(A_{0,1} - A_{1,0})^2$.
With the preceding expression:
$\frac{\partial \mathcal{L}}{\partial A_{1,0}} = -(A_{0,1} - A_{1,0}) = -F_{10} = -E_x = F^{10}$.
This is obtained by the general expression, Eq. (10.22) indeed. However, I needed with lowering the indices a change from $+$ to $-$.
For completeness, Eq. (10.22):
$\frac{\partial \mathcal{L}}{\partial A_{\mu,\nu}} = F^{\mu\nu}$.
I conclude, to derive this expression mixed indices and space indices need to be analysed separately.
On page 346, the result of the Euler-Lagrange equation for empty space is presented.

10.4 Lagrangian with Nonzero Current Density

The problem to solve is how to include J^μ, in the Lagrangian.
To this end, Susskind transformed the action integral by inferring gauge-transformation, page 348.
A gauge invariant is created with the continuity equation, Eq. (10.25).
On pages 348-350, effectiveness of this creation has been proved.
The resulting equation of motion is presented in Eq. (10.28).

Lecture 11 Fields and Classical Mechanics

11.1 Field Energy and momentum

"Energy is a conserved quantity, and it's carried by electromagnetic waves."
"Electromagnetic fields also carry momentum,……"

11.2 Three kinds of momentum

11.2.1 Mechanical Momentum

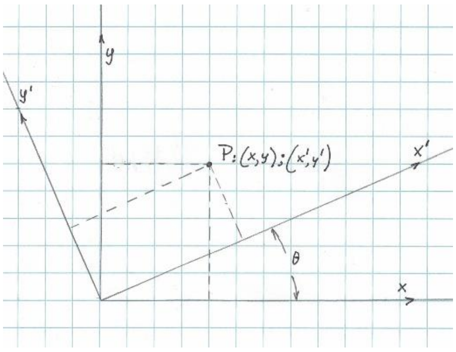
The relativistic mechanical momentum is given in Eq. (11.3)

11.2.2 Canonical Momentum

"Canonical momentum is an abstract quantity that can apply to any degree of freedom."
Instead of $\mathcal{L}$, Susskind started using L.
At the bottom of page 357, we encounter a familiar Lagrangian, Eq. (4.7) with $V(\phi) = 0$.

11.2.3 Noether Momentum

"*Noether momentum is related to symmetries".*
On page 358, Susskind presented translation symmetry.
Then, a rotation about the origin in two dimensions is presented, Eq. (11.7).
Note: this transformation is presented in Susskind (2), pages 134-135.
This rotation is also shown on page 49, Figure 1.10.



So:
$x' = x\cos\theta + y\sin\theta$,
and
$y' = -x\sin\theta + y\cos\theta$.
Then, with $\theta \longrightarrow \epsilon$, a very small angle, we have
$x' = x + y\epsilon$,
and
$y' = -x\epsilon + y$.
Now, with the representation of the Lagrangian on page 356 and $V(x) = 0$:
$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$,
after a small rotation $L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \Longrightarrow L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + O(\epsilon^2)$.
Page 359): *"A transformation of coordinates that does not change the Lagrangian is called a symmetry operation".*
Note: In Eq.(11.8) $\delta q_i \Longrightarrow \delta_i$, see Eqs. (11.5) and (11.6).

11.3 Energy

"*Momentum and energy are the space and time components of a 4 vector*".
The Hamiltonian is presented, Eq. (11.11).

11.4 Field Theory

"*Field theory is a special case of ordinary classical mechanics*".
Action comes into play again.

11.4.1 Lagrangian for Fields

The coordinates or degrees of freedom are discussed.

On the pages 362-366, Susskind explained the differences between fields and classical mechanics.

11.4.2 Action for Fields

The action here is an integral over time and space, Eq. (11.13). This illustrates the use of L for the Lagrangian. $\mathcal{L}$ is now used for the Lagrangian density:

$$L = \int d^3 \mathcal{L}(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x}) .$$

11.4.3 Hamiltonian for Fields

"To understand field energy, we need to construct the Hamiltonian".

The correspondence between the classical formulation and the "field" formulation is presented at the bottom of page 367:

$$\sum_i p_i \dot{q}_i \Longrightarrow \int d^3x \Pi_\phi(x) \dot{\phi}(x) .$$

page 368: a characteristic of all field theories is "..*conserved quantities like energy, and also momentum, are integrals of density over space*".

I present here Eq. (4.7):

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 - \left(\frac{\partial \phi}{\partial y}\right)^2 - \left(\frac{\partial \phi}{\partial z}\right)^2\right] - V(\phi) .$$

Eq. (4.14) represents a simplified version with one space dimension:

$$\mathcal{L} = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial t}\right)^2 - \left(\frac{\partial \phi}{\partial X}\right)^2\right] - V(\phi).$$

For the canonical conjugate to ϕ we need:

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \Longrightarrow \Pi_\phi = \dot{\phi} .$$

Then, with Eq. (11.14) the expression for the Hamiltonian is obtained, Eq. (11.15).

11.4.4 Consequences of Finite Energy

To have configurations with finite energy, the derivatives of ϕ with respect to x need to be smooth.

11.4.5 Electromagnetic Fields via Gauge Invariances.

Reminder:

- The fields are the four components of the vector potential A_μ.
- The Lagrangian, Eq. (10.10), $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$.

Susskind elaborated on a handsome choice of Gauge. A choice to simplify the work to do.

Again: *"Gauge transformation is about adding the gradient of an arbitrary scalar S to the vector potential A_μ".*

At the top of page 372 the choice is shown. In this way the time component of the vector potential is "neutralized'. Or fixing *the gauge.*

In this way Eq. (11.16) is obtained: *the electric field is just the time derivative of the vector potential.*

The magnetic field depends on the space components of the vector potential, Eqs. (6.29) and (6.30).

Page 375, the middle: *"The canonical momenta happen to be minus the components of the* *electric field."*

What is the Hamiltonian?

In Eq. (11.22) the electromagnetic field energy, the Hamiltonian, is presented.

Momentum Density

Noether's theorem is used. In Eq. (11.23) the shifts of the magnetic fields are presented.
To derive Eq. (11.24), use has been made of the Eqs. (11.8) and (11.9). The integrand of the integral in Eq. (11.24) is the momentum density.
On page 379, Susskind showed how to turn the momentum density into a gauge invariant quantity.
Finally, Eq. (11.25) the components of momentum are obtained:

$P_n = \int dx E_m \left(\frac{\partial A_m}{\partial X^n} - \frac{\partial A_n}{\partial X^m}\right)$.

Now *a little Algebra*(Susskind) follows:

$P_1 = \int dx [E_1 (\frac{\partial A_1}{\partial X^1} - \frac{\partial A_1}{\partial X^1}) + E_2 \left(\frac{\partial A_2}{\partial X^1} - \frac{\partial A_1}{\partial X^2}\right) + E_3 (\frac{\partial A_3}{\partial X^1} - \frac{\partial A_1}{\partial X^3})] =$

$= \int dx E_2 \left(\frac{\partial A_2}{\partial X^1} - \frac{\partial A_1}{\partial X^2}\right) + E_3 (\frac{\partial A_3}{\partial X^1} - \frac{\partial A_1}{\partial X^3})] = \int dx\, (E_2 B_3 - E_3 B_2)$.

Completely similar

$P_2 = \int dx\, (E_3 B_1 - E_1 B_3)$,

and

$P_3 = \int dx\, (E_1 B_2 - E_2 B_1)$,

are derived.

So:

$\vec{P} = (P_1, P_2, P_3) = \int d^3(x) [\, (E_2 B_3 - E_3 B_2), (E_3 B_1 - E_1 B_3), (E_1 B_2 - E_2 B_1)] =$

$= \int d^3(x) \vec{E} \times \vec{B}$.

The momentum density $\vec{E} \times \vec{B}$ is called the Poynting vector, often denoted by $\vec{S}$.

11.5 Energy and Momentum in Four Dimensions

11.5.1 Locally Conserved Quantities.

"*If we think more generally, beyond charges, we can imagine for each conservation law four quantities representing the density and the flux of any conserved quantity."*

11.5.2 Energy, Momentum and Lorentz Symmetry

The unified relativistic form of energy and momentum is given in Eq. (11.38).
Question: why is Lorentz Symmetry mention in the title of this section?

11.5.3 The Energy-Momentum Tensor

At the bottom of page 387, Susskind stated: "*What I have in mind is a symmetry or invariance argument"*.
From the formulation at the top of page 388, I assume invariance to be identical with symmetry: *"The most important symmetries of electrodynamics are gauge invariance and Lorentz invariance."*
$T^{\mu\nu}$, constructed in section 11.5.2, Eq. (11.18), is given its proper name: *Energy-Momentum Tensor*.
In Eq. (11.39), T^{00} of $T^{\mu\nu}$ is presented.
"It is formed from products of two components of $F^{\mu\nu}$."

The result of this operation is shown in Eq. (11.40).
Note: some information of this operation would be appreciated. Reading the subsequent text, I conclude it to work. Not very satisfying.

The metric tensor is used:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Remark: on page 175 this metric is presented as $\eta_{\mu\nu}$. Consequently, $\eta^{\mu\nu} = \eta_{\mu\nu}$?

With Eq. (11.40):

$$T^{00} = aF^{0\sigma}F^{0}{}_{\sigma} + b\eta^{00}F^{\sigma\tau}F_{\sigma\tau}.$$

To find out about a and b use is made of Eqs. (6.41)-(6.43).
First:

$$aF^{0\sigma}F^{0}{}_{\sigma} = a\left(E_xE_x + E_yE_y + E_zE_z\right) = aE^2.$$

Then,

$$b\eta^{00}F^{\sigma\tau}F_{\sigma\tau} = -b(F^{00}F_{00} + F^{10}F_{10} + F^{20}F_{20} + F^{30}F_{30} + F^{01}F_{01} + F^{11}F_{11} + F^{21}F_{21} +$$
$$+F^{31}F_{31} + F^{02}F_{02} + F^{12}F_{12} + F^{22}F_{22} + F^{32}F_{32} + F^{03}F_{03} + F^{13}F_{13} + F^{23}F_{23} + F^{33}F_{33} =$$
$$= -b(0 - E_xE_x - E_yE_y - E_zE_z - E_xE_x + 0 + B_zB_z + B_yB_y - E_yE_y + B_zB_z + 0 +$$
$$+B_xB_x - E_zE_z + B_yB_y + B_xB_x + 0 = -b(-2E^2 + 2B^2).$$

So, with Eq. (11.39) and $aF^{0\sigma}F^{0}{}_{\sigma} + b\eta^{00}F^{\sigma\tau}F_{\sigma\tau}$, Eq. (11.41) is obtained resulting into the expression for the energy-momentum tensor in Eq. (11.42).
Next, attention is given to the most interesting components of $T^{\mu\nu}$, top of page 390. These components are T^{0n} and T^{n0}.
From Eq.(11.40) it follows $T^{\mu\nu}$ to be symmetric.
This can be demonstrated by inspection using Eq. (11.42):

$$T^{\mu\nu} = F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} .$$

For this inspection I used Eqs. (6.41)-(6.43).
On page 390 Susskind presented the matrix for $T^{\mu\nu}$:

$$T^{\mu\nu} = \begin{pmatrix} T^{00} & S_x & S_y & S_z \\ S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yx} \\ S_z & -\sigma_{zx} & -\sigma_{xy} & -\sigma_{zz} \end{pmatrix},$$

where T^{00} is given in Eq. (11.39).

Exercise 11.1: The Poynting vector

Show that T^{0n} is the Poynting vector.
We use Eq. (11.42):

$$T^{\mu\nu} = F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \Longrightarrow T^{0n} = F^{0\sigma}F^{n}{}_{\sigma} - \frac{1}{4}\eta^{0n}F^{\sigma\tau}F_{\sigma\tau},$$

where $n = 1,2,3$.
Consequently, $\frac{1}{4}\eta^{0n}F^{\sigma\tau}F_{\sigma\tau} = 0$, due to the operation of the metric.
So, we have to deal with the elements $F^{\mu\sigma}F^{n}{}_{\sigma}$ of $T^{\mu\nu}$, with Eqs. (6.42) and (6.43),

- $n = 1$: $F^{0\sigma}F^{1}{}_{\sigma} \Longrightarrow F^{00}F^{1}{}_{0} + F^{01}F^{1}{}_{1} + F^{02}F^{1}{}_{2} + F^{03}F^{1}{}_{3} \Longrightarrow E_yB_z - E_zB_y = S_x$,
- $n = 2$: $F^{0\sigma}F^{2}{}_{\sigma} \Longrightarrow F^{00}F^{2}{}_{0} + F^{01}F^{2}{}_{1} + F^{02}F^{2}{}_{2} + F^{03}F^{2}{}_{3} \Longrightarrow E_zB_x - E_xB_z = S_y$,

- $n = 3$: $F^{0\sigma}F^{3}{}_{\sigma} \Longrightarrow F^{00}F^{3}{}_{0} + F^{01}F^{3}{}_{1} + F^{02}F^{3}{}_{2} + F^{03}F^{3}{}_{3} \Longrightarrow E_x B_y - E_y B_x = S_z$.
The three components of the Poynting vector $\vec{S}$

Exercise 11.2: Some elements of the electromagnetic stress tensor.

Calculate $T^{11}(= \sigma_{xx})$ and $T^{12}(= \sigma_{xy})$ in terms of the field components (E_x, E_y, E_z) and (B_x, B_y, B_z).
Use is made of Eq. (11.40):
$T^{\mu\nu} = F^{\mu\sigma}F^{\nu}{}_{\sigma} - \frac{1}{4}\eta^{\mu\nu}F^{\sigma\tau}F_{\sigma\tau} \Longrightarrow T^{\mu\nu} = F^{1\sigma}F^{1}{}_{\sigma} - \frac{1}{4}\eta^{11}F^{\sigma\tau}F_{\sigma\tau}$.
First, the second term in the preceding expression:
$-\frac{1}{4}\eta^{11}F^{\sigma\tau}F_{\sigma\tau} = -\frac{1}{4}(-2E^2 + 2B^2)$.
Next, $F^{1\sigma}F^{1}{}_{\sigma}$:
$F^{1\sigma}F^{1}{}_{\sigma} = F^{10}F^{1}{}_{0} + F^{11}F^{1}{}_{1} + F^{12}F^{1}{}_{2} + F^{13}F^{1}{}_{3}$.
With Eqs. (6.42) and (6.43):
$T^{11}(= \sigma_{xx}) = -E_x^2 + 0 + B_z^2 + B_y^2 - \frac{1}{2}(-E^2 + B^2) = \frac{1}{2}E^2 - E_x^2 - \frac{1}{2}B^2 + B_y^2 + B_z^2$.
Now, $T^{12}(= \sigma_{xy})$. The metric does not contribute With Eq. (11.40):
$F^{1\sigma}F^{2}{}_{\sigma} = F^{10}F^{2}{}_{0} + F^{11}F^{2}{}_{1} + F^{12}F^{2}{}_{2} + F^{13}F^{2}{}_{3}$.
With Eqs. (6.42) and (6.43):
$T^{12}(= \sigma_{xy}) = -E_x E_y - B_y B_x$.

11.6 Bye for Now

Some wise guy said,
Outside of a dog, a book is a man's best friend. Inside of a dog it is too dark to read.

Groucho Marx, Aristotle, or Jim Brewer? That is the question.

Appendix A Magnetic Monopoles

What would a magnetic monopole be if there were such things?
With this question Susskind started this chapter.
This is good reading and some rehearsal of chapter 9.
Dirac explained in 1931 how one could "fake" a monopole.
On page 399, Susskind explained how to make the solenoid exceedingly long and thin.
I liked to show you a special stretched solenoid. However, I do not want to breach copyrights.[1]

Exercise A.1 About the EMF for a monopole

Derive Eq. (A.13), based on Eq. (9.18).
See section 9.2.4 Faraday's Law.
Bottom of page 316: "*Let's explore this EMF by using Stokes's theorem, Eq. (9.4)*".
Using Eq. (9.10):

$$\int (\nabla \times \vec{E}) \cdot \hat{n}\, dS = \oint \vec{E} \cdot dl = -\int \frac{\partial \vec{B}}{\partial t} \cdot \hat{n} dS = -\frac{d}{dt}\int \vec{B} \cdot \hat{n} dS \ .$$

[1] I refer to The Economist. There KAL's cartoon displayed a stretched solenoid. A very special one. (The Economist January 16th 2021).

$\int \vec{B} \cdot \hat{n} dS \Longrightarrow$the flux through the string ϕ .
So,
$\oint \vec{E} \cdot dl = \frac{d\phi}{dt}$.
Furthermore,
$\oint \vec{E} \cdot dl = E 2\pi r \Longrightarrow E = \frac{\dot{\phi}}{2\pi r}$.

The quantum mechanical detour in this Appendix creates the appetite for more [Susskind and Friedman (1)].

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