Archimedes' Lever, Trisectrice of an angle by Geometrical Iteration

Updated 2021-06-10, Edited and § 6.11: direct geometrical iteration of $\beta/3$ for $2\pi/3 < \beta < 2\pi$

Give me a point outside a circle and a lever …….

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Abstract

To trisect an arbitrary angle with compass and unmarked ruler(straightedge) in a way similar as used for the bisectrice is impossible.
In this paper the method of geometrical iteration for trisection is used. This trisection by iteration can be done with the same accuracy as the construction of the bisectrice: within the accuracy of the line width of a pencil and pencil-compass.

§1 Introduction

The problem is about decomposing an angle in three equal parts with help of pencil-compass and an unmarked straightedge, unmarked ruler: the trisection. An old Greek problem.
To succeed, I paraphrase Archimedes:
"*Give me a point outside a circle and I will breakdown an angle inside the circle into three equal angles*", or something like that.
Well, to decompose an angle into three equal parts the Greeks needed a ruler with two marks. The two marks are the length of the radius of the circle apart (Noordzij).
So, the question is: Is it possible to get rid of the two marks on the ruler and to decompose an angle with compass and an unmarked ruler?
I will show the possibility by geometrical iteration.
A pencil-compass in a fixed adjustment, the radius of the circle, has been used.

§2 Decomposition using Archimedes' Lever

In Fig.1 below, the possibility is demonstrated.

- Draw a circle with radius R and centre M.
- Draw, arbitrarily, l_1 through M cutting the circle in C.
- Draw, arbitrarily, l_2 through C cutting the circle in A.
- Draw a circle with radius R and centre A.
  This circle cuts l_2 in B.
- Draw l through M and B in the meantime creating β and in drawing on creating α.
- Use the propositions of isosceles triangles for ΔACM and ΔABM.
- The compass is in a fixed adjustment: the radius of the circle $R = AB$.

Hence $\beta = 3\alpha$.
The question: did we create β first and after that α?
Is this a chicken and egg problem, philosophically speaking?
Well, in the way l has been drawn, we found β in the first place and afterwards α . I consider this to be futile.

However, there is more to learn from the (de)composition in Fig.1.

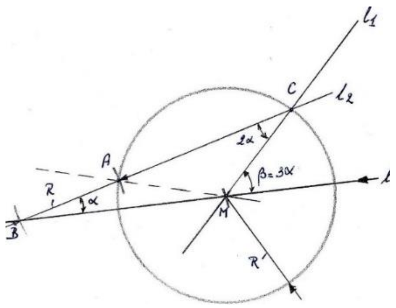

Figure 1 Construction of the decomposition

For $\beta \to 0$, $\alpha \to 0$.
Resulting into a position for point B on the diameter at a distance R from the circle. This indicated to use R as the fixed position for the compass in the iteration process.
What about increasing β? Well, at $\beta = 3\pi/4$, the points A and C collapse into a single point (C' not indicated in Fig.1) on the circle. The ΔAMC reduces into a single line through M.
$\alpha = \pi/4$, and the tangent at C' is found.
For $\beta = 3\pi/4$, $\beta = 3\alpha$ is obtained in an almost trivial way. Since, by constructing $\beta = 3\pi/4$, $\alpha = \pi/4$ is constructed. What about $3\pi/4 < \beta < \pi$?[1]
In addition, the composition in Fig.1 initiated the solution of the real trisection problem: Geometrical Iteration with unmarked ruler and the compass fixed with radius R.
In §3 Geometrical Iteration is explained and demonstrated for randomly chosen angle $\beta \leq 3\pi/4$.

§3 Decomposition with Archimedes'Lever and Geometrical Iteration.

Another way to decompose β is by geometrical iteration up to an accuracy of the line width of your pencil-compass and pencil. Well, that is not too bad to say the least. By decomposing an angle into two equal parts, the bisectrice, the accuracy is also of the line width of your pencil and pencil-compass
The 6 steps of the iteration are shown in Fig. 2.

[1] This will be discussed in § 6.1

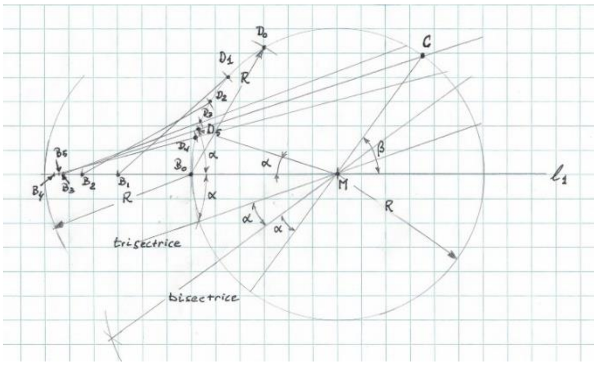

Figure 2 Geometrical Iteration for $\beta < \pi/2$

§ 3.1 The iteration proces.

The aim of the geometrical iteration to bring C, D_i and B_i on a single line , i.e., within the line width of a pencil and pencil-copmpass.
How to start?
Construct, with fixed compass at radius R, the middle of MB_0 and $(MB_0 + R) \rightarrow B_1$.
Draw a circle with radius R and centre B_1.
In this way a new point on the circle M is found → D_1 below D_0 .
Check whether C, D_1 and B_1 are on the same line within the line width of the pencil. If not, continue.
The next iteration is to construct the middle of MB_1 and $(MB_0 + R) \rightarrow B_2$. Draw a circle with radius R and centre B_2 and a new point on the circle M is found → D_2.
Check whether C, D_2 and B_2 are on the same line within the line width of the pencil. If not, continue.
Similarly, D_3 is obtained. C, D_3 and B_3 are still not on the same line. Draw the line B_3D_3. Establish this line to cut the circle M just above C.
Next, with the line B_4D_4 the circle M is crossed just below C. Within the line width of a pencil another iteration step is taken. Now, D_4 lies below the line CB_4. The next iteration step must be taken in the opposite direction[2]: Construct the middle of MB_4 and $MB_3 \rightarrow B_5$.
Within accuracy of the line width of a pencil B_5, D_5 and C are on the same line.
That is the end of the iteration proces.
We have obtained $\alpha = \beta/3$ within the accuracy of pencil width(thickness).
Summarize:
In Fig.2 we draw a circle with radius R and choose arbitrarily an angle $\beta \leq 3\pi/4$ to be decomposed into three equal angles α .
Then, choose on l_1 point B_0 on the circle M. Draw a circle with radius R and centre B_0 . In this way D_0 is obtained.

[2] This case, reversing the direction of iteration, is analysed § 4.

To find α we have to move D_i along the circle in such a way that C, D_i and B_i are on the same line (See Fig. 1- point A). This is done by, what I nominate, geometrical iteration.

In fig.2a below, the last step B_5 is oncemore demonstrated. This is done to complete the trisectrice and show $\alpha = \beta/3$, also using the bisectrice, asof β.
So, draw a circle with radius R and centre B_5. This circle cut the lower part of the circle with radius M. In this way α is constructed in the lower part. By drawing the line through M and construct the bisectrice, β has been divided in three equal parts within the line width of the pencil and pencil-compass used.

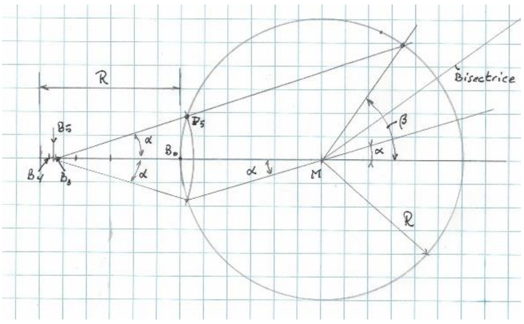

Figure 2a Detail Fig.2 Last step of iteration

§ 4 The feasibility of Geometrical Iteration

§ 4.1 A numerical example: the trisectrice of $\pi/2$

The geometrical iteration of §3 is demonstrated for the trisectrice of $\pi/2$.[3]

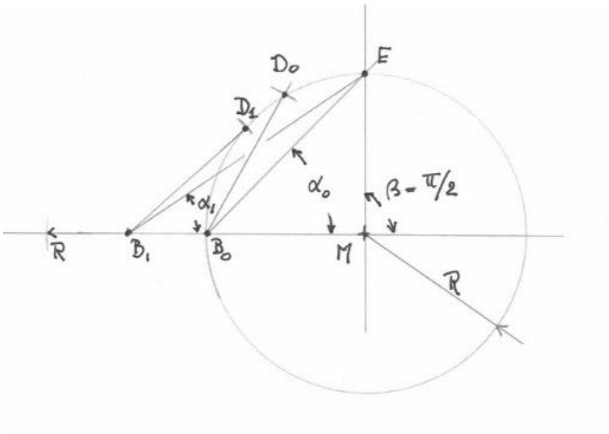

Figure 3a Numerical Example The Trisectrice of π/2

D_0 is to the left of E in the upper part of the circle M.
In Fig.3a, I showed the first two iterationsteps.

[3] To find an angle equal to $\pi/6$, an equilateral triangle can be used. See § 6.3, direct construction.

$\tan\alpha_0 = \frac{EM}{B_0M} = \frac{R}{R} = 1$, first iteration step,

$\tan\alpha_1 = \frac{EM}{B_0M+R/2} = \frac{EM/R}{B_0M/R+1/2} = \frac{1}{1+2^{-1}} = 0.667$, second iteration step

Next, α_2 not shown in Fig.3(to prevent too much cluttering):

$\tan\alpha_2 = \frac{1}{1+2^{-1}+2^{-2}}$.

and B_2 (not shown) is to the left of B_1.

The theoretical value:

$\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \cong 0.577$.

However, $\tan\alpha_2 = \frac{1}{1+2^{-1}+2^{-2}} \cong 0.571$, third iteration step.

Consequently, the direction of iteration has to be changed and the middle of B_2B_1(not shown) constructed with the compass giving B_3(not shown).

The – sign for the next step:

$\tan\alpha_3 = \frac{1}{1+2^{-1}+2^{-2}-2^{-3}} \cong 0.615$,

Then the middle of B_2B_3(not shown) is constructed giving B_4(not shown). The fifth iteration.

It is no longer possible to discern the next step giving,

$$\tan\alpha_6 = \frac{1}{1+2^{-1}+2^{-2}-2^{-3}+2^{-4}+2^{-5}+2^{-6}} = \frac{1}{1+2^{-1}+2^{-2}+2^{-3}+2^{-4}+2^{-5}+2^{-6}-2\cdot 2^{-3}} =$$

$$= \frac{1}{2[1-\left(\frac{1}{2}\right)^6-\left(\frac{1}{2}\right)^3]} \cong 0.576 ,$$

where use has been made of the summation of a geometrical series.

This numerical example demonstrates the geometrical iteration.

Keep in mind: it's about the demonstration of the accuracy of the geometrical iteration method. It appears, not shown in Fig.3a, that, within the accuracy of the unmarked ruler, the line width of pencil-compass and pencil, it's already difficult to distinghuish $\tan\alpha_2 \cong 0.571$ and the theoretical value $\tan\frac{\pi}{6} \cong 0.577$.

As mentioned in §3, when in the geometrical iteration process D_4, Fig.2, results in a point below C on the circle M, the iteration process must change direction in the next iteration step. In this way D_5 is found and the angle $\alpha = \beta/3$ within the line thickness of the pencil.

In the numerical example of this section I could not have found the (-2^{-3}) iteration step with the accuracy of the line thickness of my pencil. In other words, I did not reverse the geometrical iteration process and stopped already after I found α_2. The difference between $\tan\alpha_2 = 0.571$, and the theoretical value $\tan\frac{\pi}{6} = 0.577$, is too small to notice with the geometrical iteration process. This demonstrates the feasibility of geometrical iteration.

§ 4.2 The turning-point of geometrical iteration for $\alpha = \beta/3$ and $0 < \beta < \pi/2$

In Fig. 3b, the first two iteration steps are shown.

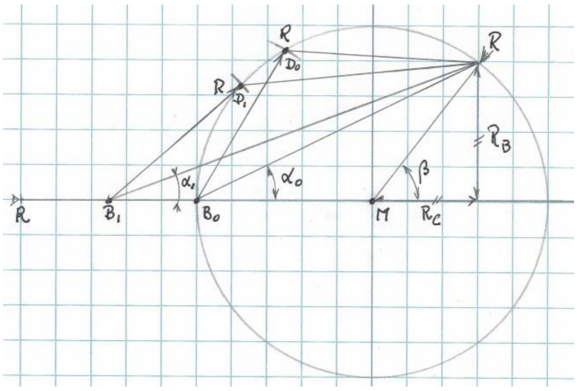

Figure 3b Feasibility of Iteration

In Fig.3b the first two steps, 0 and 1, of the geometrical iteration are shown.

$$\tan\alpha_1 = \frac{R_B}{B_1M+R_C} = \frac{R_B}{R+\frac{R}{2}+R_C} = \frac{R_B/R}{1+\frac{1}{2}+R_C/R} = \frac{\sin\beta}{1+\frac{1}{2}+\cos\beta}.$$

So, with the next iteration steps, including at least one change of sign in the iteration process, represented by the term $-\left(\frac{1}{2}\right)^k$, we have

$$\tan\alpha_n = \frac{\sin\beta}{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots-2(\frac{1}{2})^k+\cos\beta} = \frac{\sin\beta}{2[1-\left(\frac{1}{2}\right)^n-(\frac{1}{2})^k]+\cos\beta},$$

where $\{k \in \mathbb{P} | k > 1\}$.

Let us look into the case where $\alpha_n \to \frac{\beta}{3}$.

So,

$$\tan(\tfrac{\beta}{3}) = \frac{\sin\beta}{2[1-\left(\frac{1}{2}\right)^n-(\frac{1}{2})^k]+\cos\beta} \to 2\left[1-\left(\frac{1}{2}\right)^n-(\tfrac{1}{2})^k\right] = \frac{\sin\beta}{\tan(\frac{\beta}{3})} - \cos\beta = \frac{\sin(\frac{2\beta}{3})}{\sin(\frac{\beta}{3})} =$$

$$= 2\cos(\beta/3) \to \left(\frac{1}{2}\right)^n = 1-(\tfrac{1}{2})^k - \cos(\beta/3).$$

For $n \to \infty$, using special pencil-compass and pencil,

$$\cos(\beta/3) = 1-\left(\frac{1}{2}\right)^k.$$

For

$$k = 0 \to \frac{\beta}{3} = \frac{\pi}{2} + m\cdot\pi\,,$$

with $\{m \in \mathbb{N}\}$.

$$k = 1 \to \frac{\beta}{3} = \frac{\pi}{3}.$$

We obtain a relation between k, the number related to the turning-point and β.

$$\cos(\beta/3) = 1-\left(\frac{1}{2}\right)^k \to k = -\frac{\ln(1-\cos(\frac{\beta}{3}))}{\ln 2}.$$

In § 4.1 we analysed the numerical example $\beta = \frac{\pi}{2}$.

So,

$\cos(\beta/3) = \frac{\sqrt{3}}{2}$.

Plug $\cos(\beta/3) = \frac{\sqrt{3}}{2}$,

into

$k = -\frac{\ln(1-\cos(\frac{\beta}{3}))}{\ln 2} \to k = 2.9$.

This value of $k = 2.9$. proves the turning-point of the geometrical iteration to be between the third $k = 2$ and the fourth iteration $k = 3$. Consequently, we have to reverse geometrical iteration for $k = 3$. Not indicated in Fig.3b.

Another example $\beta = \frac{\pi}{4}$.

Then, you will find

$k = 4.89$,

and the turning-point lies between the 5th , $k = 4$, and the 6th , $k = 6$, iteration.

Note: D_0 is to the left of the point where β intersects the upper part of the circle.

The question which arises is: Just one Turning Point? If not, is k a meaningful number? I pay attention to the subject matter in § 6.9: *Interlude on Turning Points.*

§ 4.3 Some further observations on the accuracy of geometrical iteration.

We use $k = 3$.

Well,

$\frac{\sqrt{3}}{2} \cong 1 - \left(\frac{1}{2}\right)^3$,

establishing the accuracy of geometrical iteration.

Remember, the fith iteration in § 4.1 was hardly discernable from the fourth iteration. In stead of for $n \to \infty$, use $n = 5$ and $k = 3$.

Then,

$\cos(\alpha) = 1 - \left(\frac{1}{2}\right)^5 - \left(\frac{1}{2}\right)^3 = \frac{27}{32} = 0.844$,

and the theoretical value

$\cos(\beta/3) = \frac{\sqrt{3}}{2} = 0.866$.

So,

$\frac{0.866-0.844}{0.866} = 0.025$.

There is more about accuracy. Note, the iterated α > the theoretical $\beta/3$.

We may assume the difference between the iterated angle and the theoretical value to be a small fraction of $\beta/3$ or α . Let us call the difference Δ . So $\frac{\Delta}{\beta/3} \ll 1$.

Then,

$\cos(\alpha) = \cos(\frac{\beta}{3} + \Delta) = \cos(\beta/3)\cos\Delta - \sin\Delta\sin\frac{\beta}{3}$.

Up to $O(\Delta^2)$:

$\cos(\alpha) = \cos(\frac{\beta}{3} + \Delta) = \cos(\beta/3) - \frac{\Delta}{2} \to \frac{\Delta}{2} = \cos(\beta/3) - \cos(\alpha) = 0.022$,

and $\Delta = 0.044$.

Knowing the approximation of Δ , we can estimate the order of magnitude of the line width of a pencil and pencil-compass.
The radius R of the circle for the geometrical iteration is about $5 \cdot 10^{-2}$ m.
Hence, the arclength
$R\Delta \cong 2 \cdot 10^{-3}$m.
I consider the arclength representative of the accuracy. The accuracy is of the order of 10^{-3}m.

§ 5 Decomposition without Archimedes'Lever

Now something totally different.
An approach without Archimedes' Lever presented in Fig. 4.
Composing or decomposing is again the question.
The idea here is to start with an angle 2/3 as large as any angle you want to divide into three equal angles.

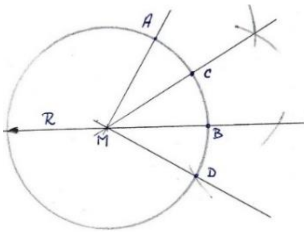

Figure 4 Decomposition?

Draw a circle with radius R and centre M.
Draw arbitrarily $\angle AMB$.
Construct the bisectrice of $\angle AMB$: MC.
MB is the bisectrice of the angle of which $\angle CMB$ is one halve.
Construct the other halve of $\angle CMB$ knowing MB to be the bisectrice., mirroring $\sphericalangle CB$. You obtain D on the circle M.
By drawing the line MD you have constructed with one stroke of your pencil $\angle AMD$ and the three angles $\angle AMC = \angle CMB = \angle BMD = \frac{1}{3}\angle AMD$.
A construction within the accuracy of the line width of your pencil.
Is this a composition or a decomposition? That is the question.

A slightly different approach for the trisectrice can be chosen.
This is shown in Fig. 5 below, page 6.
Now you start with an angle 4/3 as large as any angle you want to decompose into three equal angles: the trisectrice.
It represents a "variant" of the construction, illustrated in Fig. 5.

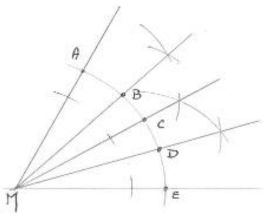

Figure 5 The trisectrice, an alternative.

Choose any angle $\angle AME$ and construct the bisectrice of this angle: MC.
Construct the bisectrices of $\angle AMC(= MB)$ and $\angle CME(= MD)$.
In this way you will obtain the trisectrice of $\angle AMD$ and $\angle BME$.
Not so much an Archimedean lever but Columbus 'egg.

§ 6 Discussion and Conclusions

When it comes to make a choice, I prefer decomposition by geometrical iteration, § 3.
There you start with any given angle and construct the trisectrice of this angle with an accuracy of your pencil thickness. Note: β defined in Fig.1.
For $\pi/2 < \beta < 2\pi/3$, the iteration procedure is the same as dealt with in §3. Nevertheless, I will demonstrate the iteration in § 6.2
For $2\pi/3 < \beta < 3\pi/4$, the procedure is again the same as dealt with in §3. Closer to $3\pi/4$, the iteration procedure becomes a bit more difficult, to say the least. Constructing within the line thickness of a pencil the geometrical iteration stops being close to the tangent position. The case $\beta = 3\pi/4$ is discussed in §2.
Well, there is more to say about the geometrical iteration. Let us start with $\beta \sim \pi/20$.

§ 6.1 Geometrical Iteration for $\beta \ll \pi/2 \Longrightarrow \beta \sim \pi/20$.

Let's start to investigate the accuracy of the geometrical iteration. Choose $\beta \sim \pi/20$.
Then it appears, after the first iteration, iterations can be left out with sufficient accuracy as shown in Fig.6.

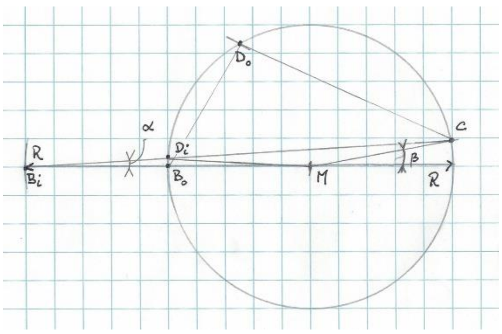

Figure 6 Geometrical Iteration for β~π/20

The first iteration step is shown: $B_0D_0(= R)$. Furthermore, the final step B_iD_i. Not all the details are shown, because most of the iteration steps are awfully close to B_i and $B_0B_i \approx R$.
Consequently, you can draw the line B_iC, with the first iteration step.

Note about the turning point: $k = -\frac{\ln\left(1-\cos\left(\frac{\beta}{3}\right)\right)}{\ln 2} = -\frac{\ln\left(1-\cos\left(\frac{\pi}{60}\right)\right)}{\ln 2} = 9.51$, (see § 4.2).
The turning point lies between the 10th and the 11th iteration step. Obviously, impossible to show in the above picture. Furthermore, D_0 is to the left of C in the upper part of the circle M.

§ 6.2 Geometrical Iteration for $\beta = \pi/2$.

This case is dealt with in § 4.1 by using iteration.

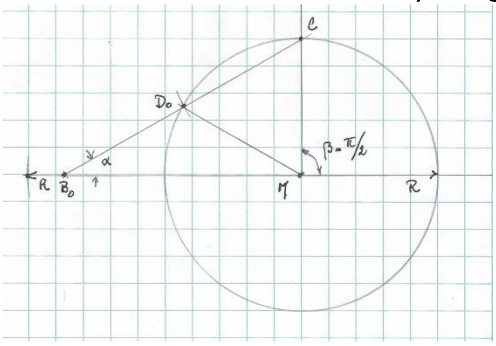

Figure 7 Geometrical Iteration for $\beta = \pi/2$

Here we show the direct construction. As usual, we start with the circle M. Draw a circle with radius R and centre C, Fig.7, on circle M. This circle, C, cut the circle with centre M in $\mathrm{D_0}$. $\angle\mathrm{MCD_0} = \frac{\pi}{3}$. Draw the line through C and $\mathrm{D_0}$. Consequently, $\angle\mathrm{D_0B_0M} = \alpha = \frac{\pi}{6}$.
Note: $|\mathrm{D_0B_0}| = \mathrm{R}$ and $\mathrm{D_0}$ is to the left of C on the upper part of circle M.

§ 6.3 Geometrical iteration $\pi/2 < \beta < 2\pi/3$ and $\beta = 2\pi/3$

Now the result of the first iteration D_0 coincides with $\beta = \frac{2\pi}{3}$, and D_0 is to the left of C on the upper part of circle M
We will analyse $\pi/2 < \beta < 2\pi/3$; $\beta \sim \frac{5}{9}\pi$.
The first three iteration steps are shown in Figure 8.

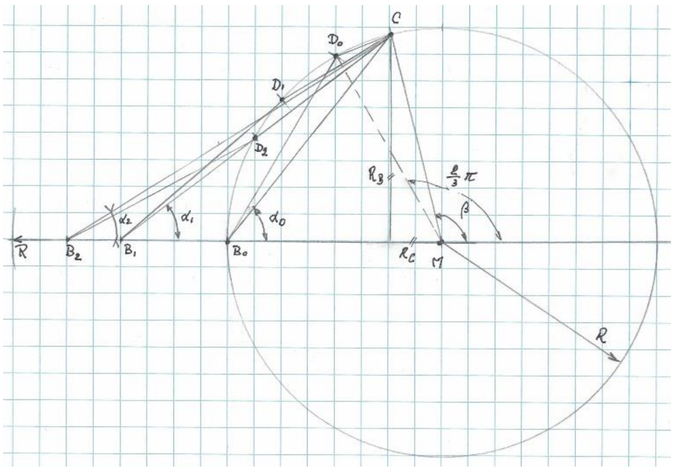

Figure 8. $\pi/2 < \beta < 2\pi/3$

The third iteration step resulted into D_2 below the line B_2C. So, the turning-point for iteration lies between the third and the fourth iteration step. Consequently we have to

construct the fourth iteration step in the middle of B_2B_1. In this way we find α_3, not shown in Figure 8 in order to prevent cluttering.
Since we have to deal with a turning point, we have

$$\tan\alpha_3 = \frac{\sin\beta}{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}-2(\frac{1}{2})^3+\cos\beta} = \frac{\sin\beta}{2[1-\left(\frac{1}{2}\right)^4-\left(\frac{1}{2}\right)^3]+\cos\beta},$$

where $\tan\alpha_3$ is derived like $\tan\alpha_n$ in § 4.2 *The turning-point of geometrical iteration for* $\alpha = \beta/3$.
As you may conclude from Figure 8, the fourth iteration step produces $\alpha = \frac{\beta}{3}$, within the required accuracy.
With the results of § 4.2: $k = -\frac{\ln\left(1-\cos\left(\frac{\beta}{3}\right)\right)}{\ln 2} \to k = 2.548$, between the third and fourth iteration. In addition to $\beta \sim \frac{5}{9}\pi$, I present here the geometric iteration for $\beta = \frac{2}{3}\pi$. See Figure 8a below.

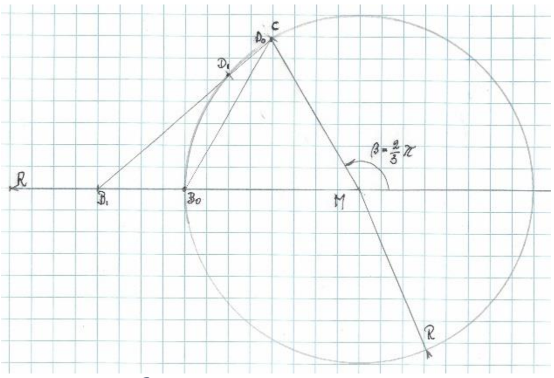

Figure 8a $\beta = \frac{2}{3}\pi$

The first iteration step coincides with β, D_0 and C. With the second iteration step $\frac{\beta}{3}$ is obtained within the line width of the pencil.

§ 6.4 Geometrical iteration $2\pi/3 < \beta < 3\pi/4$

For $2\pi/3 < \beta < 3\pi/4$, D_0 resulting from the first iteration step lies to the right of C, as indicated Figure 9. The iteration procedure is the same as in the procedure in § 3.
D_0, the result of the first iteration step lies to the right of point C, representing β, on the circle M. In this example $\beta \sim \frac{41}{60}\pi$.

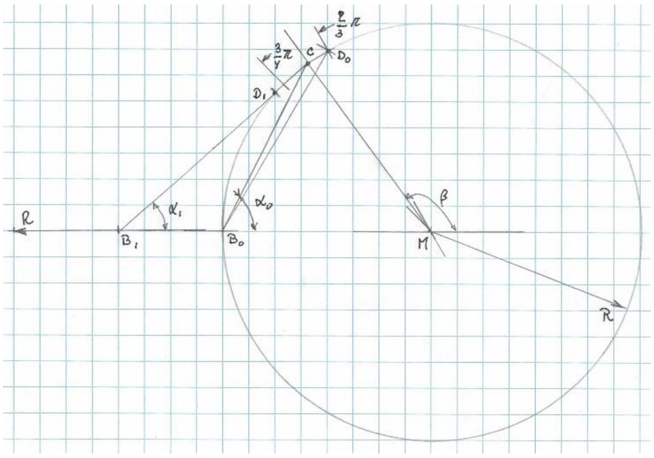

Figure 9: $2\pi/3 < \beta < 3\pi/4$

The second iteration step produces $\alpha_1 = \alpha = \beta/3$, within the accuracy of the line width of a pencil.

With the results of § 4.2: $k = -\frac{\ln\left(1-\cos\left(\frac{\beta}{3}\right)\right)}{\ln 2} \rightarrow k = 2.03$, where $k = 2$ is the third iteration step.

§ 6.5 Geometrical iteration $3\pi/4 < \beta < \pi$

With the iteration procedure of §§3 and 4, starting at B_0, Fig.10, D_0 lies to the right of C.

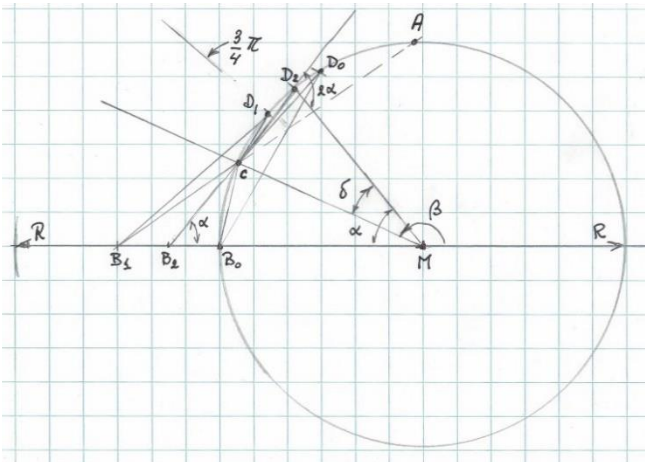

Figure 10: $3\pi/4 < \beta < \pi$

$\beta \sim 5\pi/6$.
Construct, with fixed compass at radius R, the middle of MB_0 and $(MB_0 + R) \Longrightarrow B_1$.
Draw a circle with radius R and centre B_1.
In this way a new point on the circle M is found $\rightarrow D_1$ below D_0 .
Check whether C, D_1 and B_1 are on the same line within the line width of the pencil. If not, continue. However, notice the line connecting B_1 and C cuts the circle M above D_0 in A.
So, to continue geometrical iteration, we need to reverse the iteration step. Hence, the middle of MB_0 and MB_1 is constructed $\Longrightarrow B_2$. So, instead to the left, the next step is taken to the right. The turning point lies between the first and the second iteration step.[4]

[4] The position of this turning point is analysed in § 6.6

Draw a circle with radius R and centre B_2 and a new point on the circle M is found $\Longrightarrow D_2$. Check whether C, D_2 and B_2 are on the same line within the line width of the pencil. If not, continue. Now, within the line width of a pencil C, D_2 and B_2 are on the same line.
That is the end of the iteration proces.

With the results of § 4.2: $k = -\frac{\ln\left(1-\cos\left(\frac{\beta}{3}\right)\right)}{\ln 2} \rightarrow k = 1.49$, where $k = 2$ is the third iteration step.

Did I find $\alpha = \frac{\beta}{3}$?

$\Delta B_2 D_2 M \Longrightarrow |B_2 D_2| = |D_2 M| = R.$

$\Delta C D_2 M \Longrightarrow |CM| = |D_2 M| = R \Longrightarrow \angle MCD_2 = \angle MD_2 C.$

Hence,

$\angle MCD_2 + 2\alpha = \pi,$

$\beta + \alpha - \delta = \pi.$

$\angle MCD_2 + \angle MD_2 C + \delta = \pi \Longrightarrow 2 \cdot \angle MCD_2 + \delta = \pi.$

With $2 \cdot (\angle MCD_2 + 2\alpha) = 2\pi$.

Subtract from the latter expression $2 \cdot \angle MCD_2 + \delta = \pi \Longrightarrow \delta - 4\alpha = -\pi.$

Substitute $\delta = 4\alpha - \pi$ into $\beta + \alpha - \delta = \pi \Longrightarrow \ \alpha = \frac{\beta}{3}$.

An interesting result of the above equations is: $\delta = 4\alpha - \pi$. Illustrating what I mentioned before: $\alpha \rightarrow \frac{\pi}{4} \rightarrow \delta \rightarrow 0$. The tangent appears.

Another approach, having obtained α for $\beta < \frac{3\pi}{4}$ or $\beta < \frac{\pi}{2}$ for that matter, then $\frac{\pi-\beta}{3} = \frac{\pi}{3} - \alpha = \alpha'$. To complete the construction by geometrical iteration, an equilateral triangle has to be constructed with unmarked ruler and pencil-compass to obtain $\frac{\pi}{3}$.

§ 6.6 Geometrical Iteration for $\beta \sim 19\pi/20$

An interesting case is $\beta \sim \frac{19}{20}\pi$. The construction of $\alpha = \frac{\beta}{3}$, differs completely from what we found for $\beta \sim \frac{1}{20}\pi$.

With the results of § 4.2: $k = -\frac{\ln\left(1-\cos\left(\frac{\beta}{3}\right)\right)}{\ln 2} \rightarrow k = 1.13$, where $k = 2$ is the third iteration step.

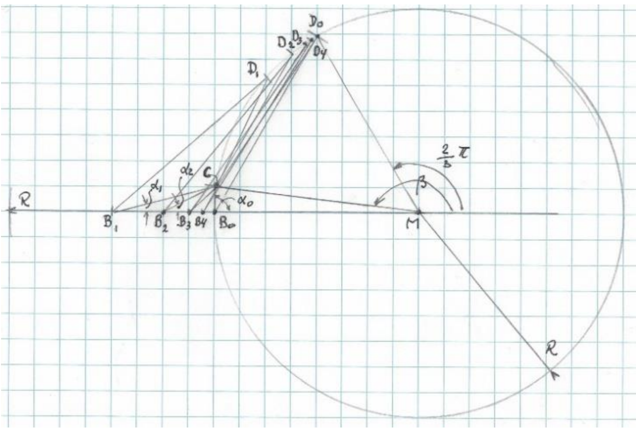

Figure 11 $\beta \sim \frac{19}{20}\pi$

Here, Figure 11, I showed five iteration steps. It created a lot of cluttering. The final step I did not show. $\alpha = \frac{\beta}{3}$ is obtained after the final step within the line width of my pencil.
The turning point of iteration lies between the first and the second iteration step.
It is difficult to discern in the above picture, however, by blowing up the details in the neighbourhood of C, we obtain the same formula for k as derived in § 4.2. Hence

$$k = -\frac{\ln\left(1-\cos\frac{\beta}{3}\right)}{\ln 2} = 1.13\ ,$$

with $\beta \sim \frac{19}{20}\pi$,
Consequently, the turning point lies between the first and second iteration, as illustrated in Figure 11.
We could have started the geometrical iteration at B_4. This point can be constructed with pencil and pencil-compass and lies at $\frac{R}{16}$ outside the circle M. Well, this needs some additional discussion, see § 6.9

§ 6.7 Geometrical Iteration for $\beta = \pi$.

The result follows from the first iteration step, actually direct construction.

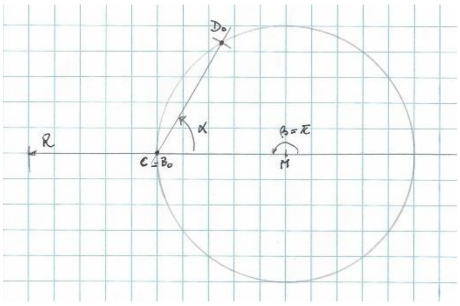

Figure 12 Geometrical Iteration $\beta = \pi$

This is a trivial case. Draw a circle with radius R and centre $C = B_0$. This circle cut the circle with centre M in D_0. $\angle MCD_0 = \frac{\pi}{3}$.

§ 6.8 Table of the results for the Turning Points of Iteration

$\beta/3$	k
$\pi/60$	9.51
$\pi/12$	4.88
$\pi/6$	2.9
$5\pi/27$	2.6
$41\pi/180$	2.03
$5\pi/18$	1.49
$19\pi/60$	1.13

§ 6.9 Interlude on Turning Points

Halfway the circle, I present some comments on turning points.
Performing the iteration, you will meet turning points using pencil, pencil-compass and unmarked ruler.
Sometimes a turning point is not discernible, due to the line width of the pencil. Sometimes it is clearly discernible. The examples can be found in the foregoing sections.
Let us analyse a couple of possibilities from a theoretical point of view.
- No turning points, see § 4.3 and $n \to \infty$:

$$\tan\beta/3 = \frac{\sin\beta}{2+\cos\beta} \to \beta = 0.$$

A trivial example.
Furthermore, examples are for those values of $\beta/3$ which can be constructed directly: (i)$\beta = \pi/3, \beta = \pi/2$, (§ 6.2), and (ii) $\beta = \pi$,(6.7). Using the bisectrice more β's can be constructed without iteration.
- One turning point (or more than one?)

The formula gives one turning point. For $\pi/2 < \beta < \pi$, this is demonstrated.
Well, we need to be more precise.
To this end, I analyse the case for $\beta \sim 19\pi/20$ further, Figure 11.
After two iteration steps, B_0 and B_1 , due to a turning point instead of continuing going to the left we need to change "course " and go to the right with the iteration. Using the notation of Figure 8, with iteration step B_4,

$$\tan\alpha_4 = \frac{\sin\beta}{1+\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}+\cos\beta}.$$

Then continuing the iteration in a theoretical way, we obtain

$$\tan\frac{\beta}{3} = \frac{\sin\beta}{3-\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\cdots\ldots\right)+\cos\beta} = \frac{\sin\beta}{3-2\left(1-(\frac{1}{2})^n\right)+\cos\beta}.$$

For $n \to \infty \to \tan\frac{\beta}{3} = \frac{\sin\beta}{1+\cos\beta} \to \cos(\beta/3) = 1 \to \frac{\beta}{3} = \frac{\pi}{3}$.
So, here we have a problem, since we want to find $\beta \sim 19\pi/60$. Hence, what I could not observe doing the geometrical iteration, there is a second turning point.
With the results of § 4.2, we need:

$$\cos(\beta/3) = 1 - \left(\frac{1}{2}\right)^k, \text{ instead of } \cos(\beta/3) = 1.$$

However, the problem encountered here, does not go away since in § 6.6 the turning point

lies between the first and second iteration for $\beta \sim 19\pi/20$. This turning point is already included in $\tan\alpha_4 = \frac{\sin\beta}{1+\frac{1}{2}-\frac{1}{4}-\frac{1}{8}-\frac{1}{16}+\cos\beta}$. This needs further analysis.
Well, let us give it a try. We could have started the geometrical iteration at B_4, since we need to be close to $\frac{\pi}{3} \sim \frac{19\pi}{20}$. This point, B_4, can be constructed with pencil and pencil-compass and lies at a distance $\frac{R}{16}$ outside the circle M.
With the notation of Figure 8, we find: $\tan\alpha_4 = \frac{\sin\beta}{1+\frac{1}{16}+\cos\beta}$. With more iteration steps,

$$\tan\alpha_n = \frac{\sin\beta}{1+\frac{1}{16}-\frac{1}{32}-\frac{1}{64}-\cdots\ +\cos\beta} = \frac{\sin\beta}{1+\frac{1}{8}-\frac{1}{16}-\frac{1}{32}-\frac{1}{64}-\cdots\ +\cos\beta} = \frac{\sin\beta}{1+\frac{1}{8}-\frac{1}{16}(1+\frac{1}{2}+\frac{1}{64}-\cdots)\ +\cos\beta} =$$

$$= \frac{\sin\beta}{1+\frac{1}{8}-\frac{1}{8}(1-\left(\frac{1}{2}\right)^n)\ +\cos\beta}.$$

For $n \to \infty \to \tan\frac{\beta}{3} = \frac{\sin\beta}{1+\cos\beta} \to \cos(\beta/3) = 1 \to \frac{\beta}{3} = \frac{\pi}{3}$.
This demonstrates a turning point lies somewhere between B_4 and B_0.
The expression for $\tan\frac{\beta}{3}$, see § 4.2, is

$$\tan\frac{\beta}{3} = \frac{\sin\beta}{1-2(\frac{1}{2})^k+\cos\beta}.$$

As we have found earlier, this results in a relation between k and $\beta/3$:

$$k = -\frac{\ln\left(1-\cos\frac{\beta}{3}\right)}{\ln 2}.$$

There is still a question to be answered: after having chosen B_4 as the starting point for the geometrical iteration, gives the preceding expression for k the turning point which we are looking for? Well, the answer is *no*. What we learned here, by choosing B_4 as a starting point for the iteration, is there exists more than one turning point.
Th consequence of this conclusion is an adjusted relation for $\tan\alpha_n$:

$$\tan\alpha_n = \frac{\sin\beta}{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots-2\left(\frac{1}{2}\right)^k-2\left(\frac{1}{2}\right)^l-2\left(\frac{1}{2}\right)^m-\cdots+\cos\beta} = \frac{\sin\beta}{2[1-\left(\frac{1}{2}\right)^n-\left(\frac{1}{2}\right)^k-\left(\frac{1}{2}\right)^l-\left(\frac{1}{2}\right)^m \ldots]+\cos\beta},$$

where $k < l < m < \cdots < n$, and $k, l, m, ..$ denoting the turning points.
For $n \to \infty$:

$$\tan\frac{\beta}{3} = \frac{\sin\beta}{2[1-\left(\frac{1}{2}\right)^k-\left(\frac{1}{2}\right)^l-\left(\frac{1}{2}\right)^m \ldots]+\cos\beta}.$$

Hence, there is no explicit relation between k and β.
These considerations do not alter the conclusion to arrive at the trisectrice, within in the line width of your pencil, by geometrical iteration. Keep in mind, by doing the iteration, we do not know the exact value of β. We need to find the trisectrice of an randomly chosen β.
When presented with a β, you can construct with your pencil-compass and the bisectrice of $\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$, etc, and compare the presented β with the constructed $\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$, etc.
By choosing a suitable starting point, the iteration can be done more efficiently.
- Alternating turning points.

$$k = -\frac{\cos(\beta/3)-1/3}{\ln 2}.$$

No examples
Next the other, lower, half of the circle.

§ 6.10 Geometrical Iteration for $\pi < \beta < 3\pi/2$.

Given a randomly chosen $\beta, \pi < \beta < \frac{3\pi}{2}$.
Use can be made of the work done for $0 < \beta < \pi$ in the upper half of the circle.
To this end we set $\beta = \pi + \beta'$, Figure 13a.
We need to construct $\frac{\beta}{3} = \frac{\pi}{3} + \frac{\beta'}{3}$.
In the § 3.1, we constructed $\frac{\beta'}{3}$ in the upper half of the circle.

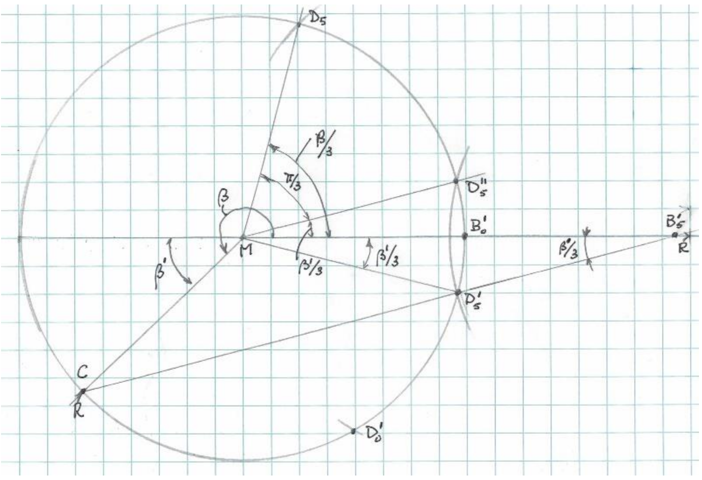

Figure 13a Geometric Iteration $\pi < \beta < \frac{3\pi}{2}$

Fig. 13a illustrates the first, D_0' and the last, D_5' iteration steps as shown in Figs.2 and 2a. In this way $\frac{\beta'}{3}$, is obtained within the line thickness of pencil-compass. Draw a circle with radius R and centre B_5' . This circle cuts the circle with centre M in D_5''. In this way, $\frac{\beta'}{3}$ is copied to the upper halve of the circle with centre M. Draw a circle with radius R and centre D_5''. In this way an angle $\frac{\pi}{3}$ is added to $\frac{\beta'}{3} \Longrightarrow \frac{\beta'+\pi}{3} = \frac{\beta}{3}$.
Now iteration without the results obtained for β in the upper half of the circle.
The iteration process is demonstrated in Figure 13b.

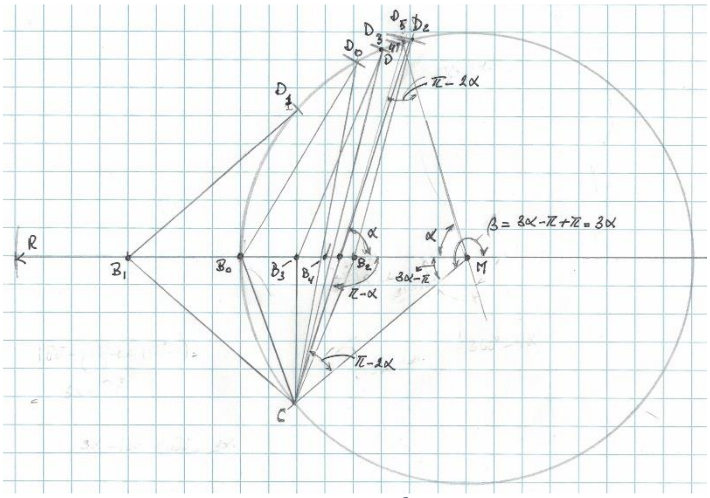

Figure 13b Geometric Iteration $\pi < \beta < \frac{3\pi}{2}$

Again, I started with drawing a circle with radius R and centre B_0. The next iteration step is B_1. From this step it follows iteration lies within the circle M. Within the line width of my pencil it appears B_5, between B_4 and B_2, to be the final discernible iteration step. The line connecting C, B_5 and D_5[5] gives the angle α. In figure 13b all the relevant angles are given→ $\beta = 3\alpha - \pi + \pi = 3\alpha$. Hence, in this way $\frac{\beta}{3}$ is constructed.

§ 6.11 Geometrical Iteration for $3\pi/2 < \beta < 2\pi$

Given a randomly chosen β, $3\pi/2 < \beta < 2\pi$.

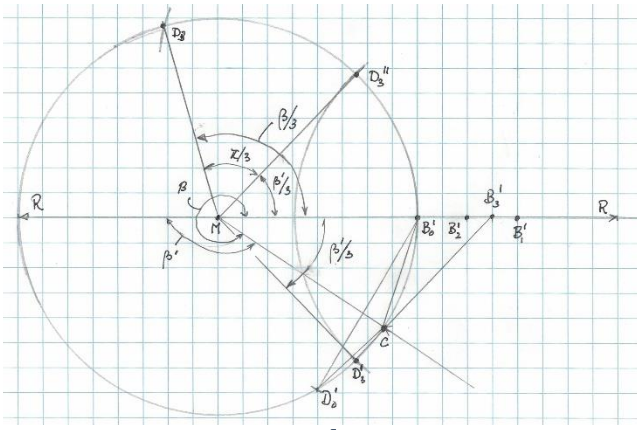

Figure 14 Geometric Iteration for $\frac{3\pi}{2} < \beta < 2\pi$

As shown in § 6.10, use can be made of the work done for $0 < \beta < \pi$ in the upper half of the circle, or by direct iteration. In Figure 14 I demonstrated the first approach. In the third iteration step B_3' , within the line thickness of the pencil, $\angle MB_3'D_3' = \frac{\beta'}{3}$, is obtained. Draw a circle with radius R and centre B_3' . This circle cuts the circle with centre M in D_3''. In this way, $\frac{\beta'}{3}$ is copied to the upper halve of the circle with centre M. Draw a circle with radius R

[5] D_4 is indicated by 4,

and centre D_3''. In this way an angle $\frac{\pi}{3}$ is added to $\frac{\beta'}{3} \Longrightarrow \frac{\beta'+\pi}{3} = \frac{\beta}{3}$. Now draw the line through D_5' and M. This line cuts the circle with centre M in D_5''. Draw a circle with radius R and centre D_5''. In this way you obtain B_2 on the line through B_0' and M. Next, draw a circle with radius R and centre B_2. This circle cuts the circle with centre M in D_5''. In this way an angle $\frac{\pi}{3}$ is added to $\frac{\beta'}{3} \Longrightarrow \frac{\beta'+\pi}{3} = \frac{\beta}{3}$.
In Figure 14a I demonstrated the other geometrical iteration procedure(direct iteration). Though β is randomly chosen, use has been made of the knowledge β to be larger than $\frac{3}{2}\pi$. This determines the starting point of the iteration procedure as shown in Figure 14a.

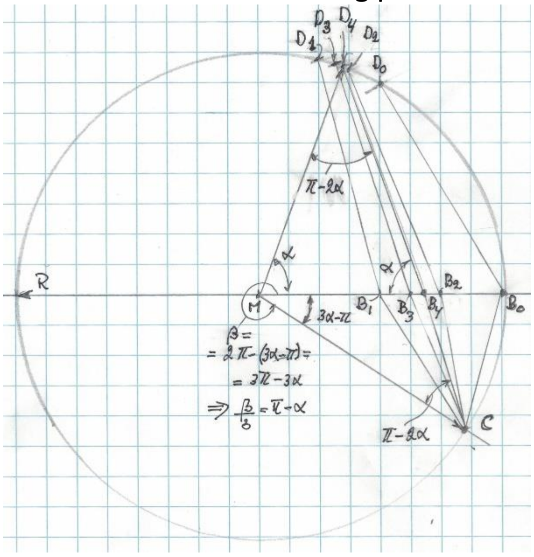

Figure 14a Geometrical Iteration for $3\pi/2 < \beta < 2\pi$

In Figure 14a the last iteration step is shown to be B_4, and $\frac{\beta}{3} = \pi - \alpha$.
Keep in mind: $B_4 D_4 = R$. This is the basis for the proof $\beta = 3\pi - 3\alpha$, as shown in Figure 14a.

§ 6.12 Conclusions

Trisection, decomposing an angle β in three equal parts can be done with the same accuracy as decomposing the angle into two equal parts. With help of:
- a pencil-compass. The compass in a fixed adjustment: the radius of the circle with centre M.
- for $\beta > \pi$, the results for $\beta < \pi$ can be used.
- the unmarked straightedge, ruler.
- a pencil.

§7 Literature

Noordzij, L., *Puzzles with my Grandchildren*, www.leennoordzij.me , 2015.

en.wikipedia.org , *Angle Trisection,* Last Edited, May 2020.