The Theoretical Minimum to Study Classical Physics

Updated 2021-06-10 Edited, A.2.3 Mathematical Proof of Kepler's Swiped Area Law, page 55 of my notes

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Remarks, Questions and Exercises.

Based on The Theoretical Minimum what you need to know to start doing physics by Susskind and Hrabovsky

Below I adopt the Lecture System of Susskind. One aspect of the layout in the book is not helpful: the use of Newton's fluxions$(\dot{x})$. So, read carefully.
Where necessary, I use the errata list as published on www.madscitech.org/tm/errata.pdf

Inhoudsopgave

Lecture 1 The Nature of Classical Physics, Page 1

1.1 What is Classical Physics? Page 1

"*The term classical Physics refers to physics before the advent of quantum mechanics*". As explained by Susskind, it is about a set of principles and rules that governs all phenomena for which quantum uncertainty is not important. It is about classical laws of physics to be deterministic.

1.2 Simple Dynamical Systems and the Space of States, page 2

A system and a closed system are defined.

Exercise 1.1 What can a closed system be?

A closed system is system that does not interact with its environment. For example, a box as a closed system not losing energy. No communication with the outside world.
However, you can do your utmost best, there will always be some leakage of information.
With an open system there exists continuous communication.

Definition: "*A system that changes with time is a dynamical system*".
Various systems are analysed. The analysis started with flipping or non-flipping a coin.
The state-space is introduced and a dynamical law, e.g., a law of motion.
The variables describing a system are called degrees of freedom.
The next system to be analysed is the six-sided die. An increasing number of states.
In the Figures 2-8, various dynamical laws are demonstrated.
Page 5: *All the basic laws of classical mechanics are deterministic.*

Exercise 1.2 A general classification of possible laws for a six-state system.

Can you think of a general way to classify the laws for a six-state system?
Well, let us analyse the die a little further.
To get a state you need to throw the die. There are obviously six possibility in each throw.
So throwing the die n times, there are 6^n possibilities: a power law.
By law you could exclude some.
For example, after throwing a 6 , the next number to be accepted is a 5, etc. Then there are $6!$ Possibilities: a permutation law.

1.3 Rules That Are Not Allowed: The minus-First Law, page 8

"*According to the rules of classical physics, not all laws are legal.*"
In classical physics a law must be deterministic and reversible.
On pages 8 and 9 irreversible and non-deterministic systems are presented.
On page 9 a reversible and deterministic system is defined. The law covering such a system is called the minus first law: conservation of information.

1.4 Dynamical Systems with an Infinite Number of States, page 10

In this section a dynamical system with an infinite number of states is discussed. An example of such a state is presented: an infinite number of discrete points along a line.
These discrete points along the line, states, are numbered: N at every n.
In addition, the states are allowable because each state has an arrow in and an arrow out, see Figures 11 and 12.
Then several rules, (1)-(5), are presented on page 11. Not all the rules are allowable.
In Exercise 1.3 the rules have been analysed. See next page.

Exercise 1.3 Allowable and non-allowable rules for states on a line

(1) $N(n+1) = N(n) + 1$.
This example is analysed in the text, page 11. At each time step $n+1$, the next state , $N(n)+1$, is marked $\Longrightarrow$ Reversible and Deterministic. Furthermore, each state has an arrow in and out.
(2) $N(n+1) = N(n) - 1$.
At each time step the next state to the left on the line, $N(n) - 1$, is marked $\Longrightarrow$ Reversible and Deterministic. Furthermore, each state has an arrow in and out.
(3) $N(n+1) = N(n) + 2$.
This example is analysed in the text on page 11, below the exercise 3.
When you start at an odd value of N, you will stay with the odd values and vice versa for the even values. However, the system is still reversible and deterministic.
The arrows are a bit different. The arrows in and out an odd state come from the former and go to the next odd state. The same procedure applies to the even numbers. Consequently, there are two infinite cycles.
(4) $N(n+1) = N(n)^2$.
Well, what does this represent? I assume it to represent: $N(n+1) = N^2(n)$.
This is an example of a non-allowable rule. There is no 'way back'.
(5) $N(n+1) = -1^{N(n)}N(n)$.
This is really something.
An example is helpful: let us take state 5. A time step gives $N(n+1) = -5$. Then, the next time step: $N(n+2) = -1^{N(n+1)}N(n+1) = (-1)^{-5} \cdot (-5) = 5$, ad infinitum. A finite cycle as illustrated in Figure 13, page 12
So, reversible and deterministic. An allowable rule.

1.5 Cycles and Conservation Laws, page 12

As illustrated in Figures 14 and 15, pages 12 and 13: *When the state-space is separated into several separated cycles, there is a memory of which cycle the system started in:* ***a conservation law.***

1.6 The Limits of Precision, page 13

"In principle we cannot know the initial conditions with infinite precision."
The concept of chaos is briefly mentioned.

Interlude 1: Spaces, Trigonometry, and Vectors, Page 15

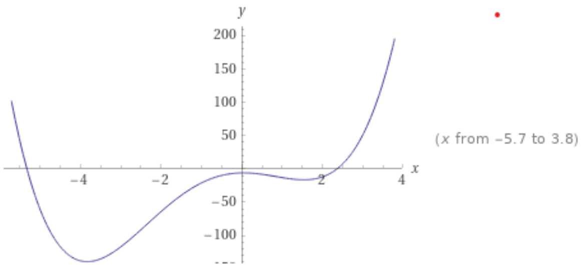
I.1.1 Coordinates, page 15

On the pages 15-18 coordinates are introduced.
Coordinates of:
- a point,
- a plane,
- the direction of time.

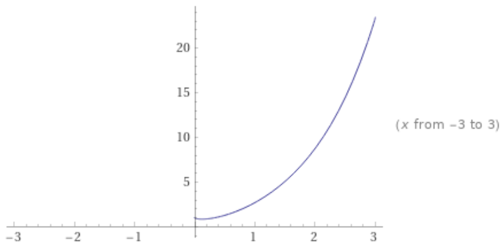
The four coordinates, x, y, z, t, define a reference frame.

Exercise I.1.1 Plotting Functions

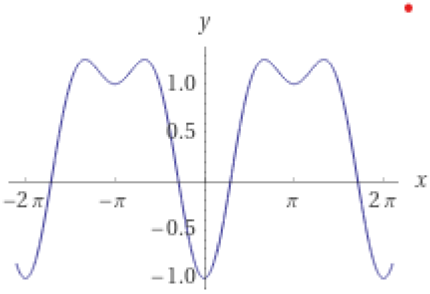
Various functions are presented. I plotted three of these functions.
Note: $x(t) = \sin^2 x - \cos x$, should read $x(t) = \sin^2 t - \cos t$, a typo.
- I used WolframAlpha to plot $f(t) = t^4 + 3t^3 - 12t^2 + t - 6$,
where f$(t) \to y$ and $t \to x$



-I used WolframAlpha to plot $\theta(\alpha) = e^{\alpha} + \alpha \ln \alpha$,
where $\theta(\alpha) \to y$ and $\alpha \to x$.



-I used WolframAlpha to plot $x(t) = \sin^2 t - \cos t$,
where $x(t) \to y$ and $t \to x$



I.1.2 Trigonometry, page 19

Note: on page 19, 1 radian$= \frac{\pi}{180°}$, should read : 1 radian$= \frac{180°}{\pi}$, a typo.
On page 23, a couple of trigonometric functions for $\alpha \pm \beta$ are summarized.
On Wikipedia, www.en.wikipedia.org , more on trigonometry can be found.

I.1.3 Vectors, page 23

A vector can be thought of as an object that has both magnitude and a direction in space.
In Figure.13 page 25, adding of vectors is demonstrated.

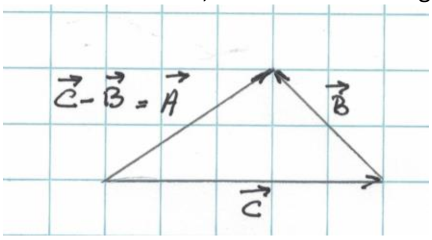
Exercise I.1.2 Vector subtraction

Work out the rule for vector subtraction.

On pages 24-25, adding vectors is discussed and illustrated in Figure 13. Similarly, vector subtraction can be analysed.

In Figure 13: $\vec{A} + \vec{B} = \vec{C}$.

Hence: $\vec{C} - \vec{B} = \vec{A}$, illustrated in the Figure below:



Exercise I.1.3 The Magnitude of a Vector

Show that the magnitude of a vector satisfies $|\vec{A}|^2 = \vec{A} \cdot \vec{A}$.

The dot product of two vectors $\vec{A}$ and $\vec{B}$ is defined as, page 27,

$\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos\theta$,

θ being the angle between the two vectors.

With $\vec{B} \equiv \vec{A} \therefore \theta = 0$.

Consequently,

$|\vec{A}|^2 = \vec{A} \cdot \vec{A}$.

Exercise I.1.4 The magnitude, the dot product, and the angle between two vectors

Let $(A_x = 2, A_y = -3, A_z = 1)$ and $(B_x = -4, B_y = -3, B_z = 2)$.

- The magnitude of $\vec{A}$ and $\vec{B}$:

$|\vec{A}|^2 = \vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2 = 14 \therefore |\vec{A}| = \sqrt{14}$.

$|\vec{B}|^2 = \vec{B} \cdot \vec{B} = B_x^2 + B_y^2 + B_z^2 = 29 \therefore |\vec{B}| = \sqrt{29}$.

- The dot product $\vec{A} \cdot \vec{B} = |\vec{A}| \cdot |\vec{B}| \cdot \cos\theta$.

So,

$\vec{A} \cdot \vec{B} = \sqrt{14 \cdot 29} \cdot \cos\theta$.

- The angle between $\vec{A}$ and $\vec{B}$.

We need one more expression, bottom page 27:

$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = -8 + 9 + 2 = 3$.

Then,

$\cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{|\vec{A}| \cdot |\vec{B}|} = \frac{3}{\sqrt{14 \cdot 29}}$,

and

$\theta = \cos^{-1} \frac{3}{\sqrt{14 \cdot 29}}$.

Exercise I.4 teaches us the dot product of two vectors being orthogonal is 0, since the angle between the two vectors $\theta = \frac{\pi}{2}$. Consequently, $\cos\theta = 0$. The projection of orthogonal vectors on each other is 0, Exercise I.1.6.

Exercise 1.1.5 About orthogonality

This exercise is a straightforward application of
$A_x B_x + A_y B_y + A_z B_z$.
Then you will find the pairs of vectors $(2, -1,3)$ and $(-3,0,2)$ are orthogonal.
Plug these numbers into $A_x B_x + A_y B_y + A_z B_z \rightarrow -6 + 0 + 6 = 0$.

Lecture 2: Motion, Page 29

2.1 Mathematical Interlude: Differential Calculus, page 29

Functions varying with time in a continuous way are dealt with.
Calculus is used, so the concept of limit is needed, pages 29-31.
Rules are presented:
- the sum rule, page 35,
- the product rule, page 35,
- the chain rule, page 35.

Exercise 2.1 Calculation of Derivatives

- $f(t) = t^4 + 3t^3 - 12t^2 + t - 6$.
$\frac{df}{dt} = 4t^3 + 9t^2 - 24t + 1$.
- $g(x) = \sin x - \cos x$.
$\frac{dg}{dx} = \cos x + \sin x$.
- $\theta(\alpha) = e^\alpha + \alpha \ln \alpha$.
$\frac{d\theta(\alpha)}{d\alpha} = e^\alpha + \ln \alpha + 1$.
- $x(t) = \sin^2 x - \cos x$? Or is it $x(t) = \sin^2 t - \cos t$, see Exercise 1.1.
Assume it to be $x(t) = \sin^2 t - \cos t$:
$\frac{dx}{dt} = 2 \sin t \cos t + \sin t$.

Exercise 2.2 The Second Derivative

The derivative of a derivative is called the second derivative and is written as $\frac{d^2 f(t)}{dt^2}$. Take the second derivative of each of the functions listed in Exercise 2.1 above.
- $\frac{d^2 f(t)}{dt^2} = 12t^2 + 18t - 24$.
- $\frac{d^2 g(x)}{dx^2} = -\sin x + \cos x$.
- $\frac{d^2 \theta(\alpha)}{d\alpha^2} = e^\alpha + \frac{1}{\alpha}$.
- $\frac{d^2 x(t)}{dt^2} = 2\cos^2 t - 2\sin^2 t + \cos t$.

Exercise 2.3 The Chain Rule

Use te chain rule to find the derivatives of each of the following functions

- $g(t) = \sin t^2 - \cos t^2$.

Use $f(t) = t^2$, then the chain rule,

$\frac{dg}{dt} = \frac{dg}{df} \cdot \frac{df}{dt}$, results into

$\frac{dg}{dt} = \frac{d\sin f}{df} \cdot \frac{df}{dt} - \frac{d\cos f}{df} \cdot \frac{df}{dt} = \cos t^2 \cdot 2t + \sin t^2 \cdot 2t = 2t(\cos t^2 + \sin t^2)$.

- $\theta(\alpha) = e^{3\alpha} + 3\alpha \ln 3\alpha \Longrightarrow$ chain rule and product rule

$\frac{d\theta}{d\alpha} = \frac{de^{3\alpha}}{d3\alpha} \cdot \frac{d3\alpha}{d\alpha} + \ln 3\alpha \cdot \frac{d3\alpha}{d\alpha} + 3\alpha \cdot \frac{d\ln 3\alpha}{d3\alpha} \cdot \frac{d3\alpha}{d\alpha} = 3e^{3\propto} + 3\ln 3\alpha + 3$.

- $x(t) = \sin^2 t^2 - \cos^2 t^2$.

Let us simplify the preceding expression by using

$\sin^2 t^2 + \cos^2 t^2 = 1$.

Consequently there remains just one expression to be differentiated:

$2 \cdot \sin^2 t^2 - 1$.

We have already some results, so:

$\frac{dx}{dt} = 2 \cdot \frac{d\sin^2 t^2}{d\sin t^2} \cdot \frac{d\sin t^2}{dt} = 4. \sin t^2 \cos t^2 \cdot 2t \cdot \cos t^2 = 8t \cdot \sin t^2 \cos^2 t^2$.

Exercise 2.4 Proof of Some Rules

The Sum Rule

- $\frac{d(f+g)}{dt} = \frac{df}{dt} + \frac{dg}{dt}$.

$\Delta(f+g) = \Delta f + \Delta g = f(t+\Delta t) - f(t) + g(t+\Delta t) - g(t)$.

Then,

$\frac{d}{dt}(f+g) = \lim_{\Delta t \to 0} \frac{\Delta(f+g)}{\Delta t} = \lim_{\Delta t \to 0} [\frac{f(t+\Delta t)-f(t)}{\Delta t} + \frac{g(t+\Delta t)-g(t)}{\Delta t}] = \frac{df}{dt} + \frac{dg}{dt}$.

The Product Rule

- $\frac{d(fg)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta(fg)}{\Delta t} = \lim_{\Delta t \to 0} \frac{(f+\Delta f)(g+\Delta g)-fg}{\Delta t} = \lim_{\Delta t \to 0} \frac{fg+g\Delta f+f\Delta g+\Delta f\Delta g-fg}{\Delta t} =$
$\lim_{\Delta t \to 0} \left[g \frac{\Delta f}{\Delta t} + f \frac{\Delta g}{\Delta t} - \Delta f \frac{\Delta g}{\Delta t}\right] = g \frac{df}{dt} + f \frac{dg}{dt}$,

since $\lim_{\Delta t \to 0} \Delta f \frac{\Delta g}{\Delta t} = 0$.

The Chain Rule

- $\frac{df}{dt} = \frac{df}{dg} \cdot \frac{dg}{dt}$.

$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t} = \lim_{\Delta t \to 0} [\frac{f(g+\Delta g)-f(g)}{\Delta g} \cdot \frac{g(t+\Delta t)-g(t)}{\Delta t}]$,

with $\frac{g(t+\Delta t)-g(t)}{\Delta t} = \frac{\Delta g}{\Delta t}$, and $\frac{f(g+\Delta g)-f(g)}{\Delta g} = \frac{\Delta f}{\Delta g}$

$\frac{df}{dt} = \lim_{\Delta t \to 0} [\frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta t}] = \frac{df}{dg} \cdot \frac{dg}{dt}$.

In the next exercise use will be made of: $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

Exercise 2.5 A few more Proofs.

- $\frac{d\sin t}{dt}=\cos t.$

$\frac{d\sin t}{dt}=\lim_{\Delta t\to 0}\frac{\Delta\sin t}{\Delta t}=\lim_{\Delta t\to 0}\frac{\sin(t+\Delta t)-\sin t}{\Delta t}=\lim_{\Delta t\to 0}\frac{\sin t\cos\Delta t+\cos t\sin\Delta t-\sin t}{\Delta t}=$

$\lim_{\Delta t\to 0}\frac{\sin t\cos\Delta t-\sin t}{\Delta t}+\lim_{\Delta t\to 0}\frac{\cos t\sin\Delta t}{\Delta t}=\sin t\lim_{\Delta t\to 0}(\frac{\cos\Delta t-1}{\Delta t})+\cos t\lim_{\Delta t\to 0}\frac{\sin\Delta t}{\Delta t}.$

With $\lim_{\Delta t\to 0}\left(\frac{\cos\Delta t-1}{\Delta t}\right)=0?$

Well, this is correct. However, knowledge is needed for the series expansion of $\cos\Delta t$. Then, we need derivatives. That is how it looks like. But look, see Courant page 48,

$\frac{1-\cos\Delta t}{\Delta t}=\frac{(1-\cos\Delta t)(1+\cos\Delta t)}{\Delta t(1+\cos\Delta t)}=\frac{1-\cos^2\Delta t}{\Delta t(1+\cos\Delta t)}=\frac{\sin\Delta t}{\Delta t}\cdot\frac{1}{1+\cos\Delta t}\cdot\sin\Delta t.$

Consequently, $\lim_{\Delta t\to 0}\left(\frac{\cos\Delta t-1}{\Delta t}\right)=0.$

Use has been made of $\lim_{\Delta t\to 0}\frac{\sin\Delta t}{\Delta t}=1$. This limit is found with help of geometry(See Courant page 48, or Chisholm and Morris page 9).

Nevertheless, I choose a different approach:

$\lim_{\Delta t\to 0}\frac{\sin(t+\Delta t)-\sin t}{\Delta t}=\lim_{\Delta t\to 0}\frac{2\cos(t+\frac{\Delta t}{2})\sin\frac{\Delta t}{2}}{\Delta t}=\lim_{\Delta t\to 0}\cos(t+\frac{\Delta t}{2})\lim_{\Delta t\to 0}\frac{\sin\frac{\Delta t}{2}}{\frac{\Delta t}{2}}=\cos t.$

$\frac{d\sin t}{dt}=\cos t.$

Note: use has been made of: $\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}.$

- $\frac{d\cos t}{dt}=-\sin t.$

$\frac{d\cos t}{dt}=\lim_{\Delta t\to 0}\frac{\Delta\cos t}{\Delta t}=\lim_{\Delta t\to 0}\frac{\cos(t+\Delta t)-\cos t}{\Delta t}=\lim_{\Delta t\to 0}\frac{-2\sin(t+\frac{\Delta t}{2})\sin\frac{\Delta t}{2}}{\Delta t}=$

$=-\lim_{\Delta t\to 0}\sin\left(t+\frac{\Delta t}{2}\right)\lim_{\Delta t\to 0}\frac{\sin\frac{\Delta t}{2}}{\frac{\Delta t}{2}}=-\sin t$

$\frac{d\cos t}{dt}=-\sin t.$

- $\frac{de^t}{dt}=e^t.$

$\frac{de^t}{dt}=\lim_{\Delta t\to 0}\frac{\Delta e^t}{\Delta t}=\lim_{\Delta t\to 0}\frac{e^{t+\Delta t}-e^t}{\Delta t}=e^t\lim_{\Delta t\to 0}\frac{e^{\Delta t}-1}{\Delta t}=?$

On page 34, Susskind writes: "*Basically, e^t is defined by the property that its derivative is equal to itself. So, $\frac{de^t}{dt}=e^t$ is really a definition*". So? In textbooks(Chisholm and Morris) I found: The derivative of the exponential function is found from the definition of the logarithm.

However, $\lim_{\Delta t\to 0}\frac{e^{\Delta t}-1}{\Delta t}=1$, is the definition of e.

Consequently,

$\frac{de^t}{dt}=\lim_{\Delta t\to 0}\frac{\Delta e^t}{\Delta t}=\lim_{\Delta t\to 0}\frac{e^{t+\Delta t}-e^t}{\Delta t}=e^t\lim_{\Delta t\to 0}\frac{e^{\Delta t}-1}{\Delta t}=e^t.$

$\frac{de^t}{dt}=e^t.$

- $\frac{d\log t}{dt}=\frac{1}{t}.$

Again, in textbooks, this differential is presented as a definition.

$$\frac{d\log t}{dt}=\lim_{\Delta t\to 0}\frac{\Delta\log t}{\Delta t}=\lim_{\Delta t\to 0}\frac{\log(t+\Delta t)-\log t}{\Delta t}=\lim_{\Delta t\to 0}\frac{\log(1+\frac{\Delta t}{t})}{\Delta t}=?$$

In textbooks(de Bruijn), you will find the following:

Set

$t=e^x\Longrightarrow x=\log t\Longrightarrow\frac{d\log t}{dt}=\left(\frac{dt}{dx}\right)^{-1}=e^{-x}=\frac{1}{t}.$

$\frac{d\log t}{dt}=\frac{1}{t}.$

Remark: the series expansion of $e^{\Delta x}=\lim_{n\to\infty}(1+\frac{1}{n})^{\Delta x\cdot n}$ could have been used with Newton's binomium.

2.2 Particle Motion, page 38

The concept of point particle is an idealization.

It is about position and velocity of a particle. A rehearsal of the section on Vectors, page 23, can be helpful.

Remark: the dot notation is, well, sometimes difficult to discern, at least in my printed copy of *The Theoretical Minimum*.

2.3 Examples of Motion, page 41

- A falling particle,
- An oscillating particle.

Exercise 2.6 A Full Cycle

How long does it take for the oscillating particle go through one full cycle of motion?
Looking at Figure 3, page 44, a full cycle equals 2π.
So, with $\sin \omega t$, a full cycle means $\omega t = 2\pi$.
Hence, $t = T = \frac{2\pi}{\omega}$.

- Particle moving with uniform circular motion.

Exercise 2.7 Orthogonality of Vectors

Show that the position and velocity vectors are orthogonal. I suppose, the authors have circular motion in mind. Let us analyse the 2-D case. Eqs.(3), page 45.
The position vector $\vec{r} = R(\cos \omega t, \sin \omega t) = \big(x(t), y(t)\big)$.
The velocity vector $\vec{v} = R\omega(-\sin \omega t, \cos \omega t) = \big(v_x, v_y\big)$.
Then,

$$\vec{r} \cdot \vec{v} = R^2\omega[-\cos \omega t \cdot \sin \omega t + \sin \omega t \cdot \cos \omega t] = x(t)v_x + y(t)v_y = 0.$$

Exercise 2.8 Velocity, Speed and Acceleration

Calculate the velocity, speed, and acceleration of the following position vectors.

- $\vec{r} = (\cos\omega t, e^{\omega t})$.

$\vec{v} = \frac{d\vec{r}}{dt} = (-\omega\sin\omega t, \omega e^{\omega t})$.

$|\vec{v}| = \omega\sqrt{\sin^2\omega t + e^{2\omega t}}$.

$\vec{a} = \frac{d\vec{v}}{dt} = (-\omega^2\cos\omega t, \omega^2 e^{\omega t})$.

The x-component oscillates between fixed values, the y-component grows to infinity.

- $\vec{r} = [\cos(\omega t - \phi), \sin(\omega t - \phi)]$.

$\vec{v} = \frac{d\vec{r}}{dt} = [-\omega\sin(\omega t - \phi), \omega\cos(\omega t - \phi)]$.

$|\vec{v}| = \omega$.

$\vec{a} = \frac{d\vec{v}}{dt} = [-\omega^2\cos(\omega t - \phi), -\omega^2\sin(\omega t - \phi)]$.

The x- and the y-components oscillate between fixed values.

- $\vec{r} = (c\cos^3 t\,, c\sin^3 t)$.

$\vec{v} = \frac{d\vec{r}}{dt} = 3c(-\sin t\cos^2 t\,, \cos t\sin^2 t)$.

$|\vec{v}| = 3c\sin t\cos t$.

$\vec{a} = \frac{d\vec{v}}{dt} = 3c(2\cos t - 3\cos^3 t, 2\sin t - 3\sin^3 t)$.

The components oscillate between fixed values.

- $\vec{r} = c[(t - \sin t)\,, (1 - \cos t)]$.

$\vec{v} = \frac{d\vec{r}}{dt} = c[(1 - \cos t, \sin t]$.

$|\vec{v}| = c\sqrt{2}\sqrt{1 - \cos t}$.

$\vec{a} = \frac{d\vec{v}}{dt} = c(\sin t, \cos t)$.

The position vector grows infinitely. The other vectors oscillate between fixed values.

Interlude 2: Integral Calculus, Page 47

I.2.1 Integral Calculus, page 47

Integral calculus has to do with sums of many tiny incremental quantities.

On page 50, *the fundamental theorem* of calculus is presented. Just below this presentation, Susskind writes: "*What it says that if $F(T) = \int f(t)\,dt$, ….* ".May be, it is a bit clearer to write: $F(t) = \int f(t)\,dt$, see Eq.(1).

The relation between integration and differentiation is clearly presented. Bottom of page 50 and top of page 51.

Exercise I.2.1 Indefinite Integrals, page 54

Determine the indefinite integral of each of the following expressions by reversing the process of differentiation and adding a constant.

- $f(t) = t^4$.

So, $f(t) = \frac{dF(t)}{dt}$.

This example is explained on page 52. I repeat a few lines.

With differentiation we learned: $\frac{dt^4}{dt} = 4t^3$. Then, $\frac{dt^5}{dt} = 5t^4$.

Hence, $\frac{dt^5/5}{dt} = t^4 \Longrightarrow t^4$ is the derivative of $\frac{t^5}{5} \Longrightarrow \int t^4\, dt = \frac{t^5}{5} + c$.

- $f(t) = \cos t$.

We know $\cos t = \frac{d \sin t}{dt} \Longrightarrow \int \cos t\, dt = \sin t + c$.

- $f(t) = t^2 - 2 \frac{d(\frac{t^3}{3} - 2t)}{dt} \Longrightarrow$

$\frac{d(\frac{t^3}{3} - 2t)}{dt} \Longrightarrow \int (t^2 - 2)\, dt = \left(\frac{t^3}{3} - 2t\right) + c$.

c is a constant to be determined.

Exercise I.2.2 From indefinite to definite integral, page 55

Use the fundamental theorem of calculus to evaluate each integral from Exercise I.2.1, with limits of integration being $t = 0$ to $t = T$.

The fundamental theorem: $\int_a^b f(t)dt = F(t)|_a^b = F(a) - F(b)$.

- $\int_0^T t^4 dt = \frac{1}{5} t^5 |_0^T = \frac{1}{5} T^5$.
- $\int_0^T \cos t\, dt = \sin t\, |_0^T = \sin T$.
- $\int_0^T (t^2 - 2)dt = (\frac{1}{3} t^3 - 2t)|_0^T = \frac{1}{3} T^3 - 2T$.

Exercise I.2.3 Velocities and Trajectories, page 55

Treat the expressions from Exercise I.2.1, as expressions for the acceleration of a particle. Integrate them once, with respect to time, and determine the velocities, and a second time to determine the trajectories. Because t is used as one of the limits of integration, the dummy integration variable $t'$is adopted. Integrate them from $t' = 0$ to $t' = t$.

I will use one example:

- $a(t) = \cos t$.

$v(t) = \int_0^t \cos t'\, dt' = \sin t'\, |_0^t = \sin t$.

$s(t) = \int_0^t \sin t'\, dt' = -\cos t'\, |_0^t = 1 - \cos t$.

I.2.2 Integration by Parts, page 55

A strong instrument doing integrals, is integration by parts.

It starts with the product rule for differentiation, Lecture 2 page 35.

Then after integration, Eq.(4), page 56 is obtained.

Remark.

In integrating the results of the product rule, I prefer a slightly different notation:

$$\int_a^b \frac{d[f(x)g(x)]}{dx} dx = \int_a^b f(x) \frac{dg(x)}{dx} dx + \int_a^b g(x) \frac{df(x)}{dx} dx.$$

Then, Eq.(4) reads:

$$f(x)g(x)|_a^b - \int_a^b f(x)\frac{dg(x)}{dx}dx = \int_a^b g(x)\frac{df(x)}{dx}dx.$$

> ### Exercise I.2.4 Integration by Parts, page 57
> Evaluate the integral $\int_0^{\frac{\pi}{2}} x\cos x\, dx$.
>
> We know $\cos x = \frac{d\sin x}{dx}$.
>
> Then, $\int_0^{\frac{\pi}{2}} x\cos x\, dx = \int_0^{\frac{\pi}{2}} x\frac{d\sin x}{dx}dx$.
>
> With $f(x)g(x)|_a^b - \int_a^b f(x)\frac{dg(x)}{dx}dx = \int_a^b g(x)\frac{df(x)}{dx}dx$,
>
> we have:
>
> $(\sin x)\cdot x|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \sin x\frac{d}{dx}x\, dx = \int_0^{\frac{\pi}{2}} x\cos x\, dx.$
>
> So, $\int_0^{\frac{\pi}{2}} x\cos x\, dx = \frac{\pi}{2}\sin\frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin x\, dx = \frac{\pi}{2} + \cos x\,|_0^{\frac{\pi}{2}} = \frac{\pi}{2} - 1.$

Lecture 3: Dynamics, Page 58

3.1 Aristotle's Law of Motion, page 58

Aristotle's law of motion according to Susskind:

The velocity of any object is proportional to the total applied force.

> ### Exercise 3.1 Aristotle's Law, page 61
> Given a force that varies with time according to $F = 2t^2$, and with the initial condition at time zero, $x(0) = \pi$, use Aristotle's law to find $x(t)$ at all times.
>
> $x(t) = \int_0^t \frac{2t'^2}{m}dt' + c \Rightarrow x(t) = \frac{2}{3m}t'^3|_0^t + c = \frac{2}{3m}t^3 + c.$
>
> With the initial condition $x(0) = \pi$,
>
> $x(0) = c = \pi.$
>
> Hence, $x(t) = \frac{2}{3m}t^3 + \pi.$

Criteria for a law, Lecture 1:

- deterministic
- reversible, the resulting new law is also deterministic.

Aristotle's law is deterministic and reversible.

Alas, Aristotle's law is wrong.

3.2 Mass, Acceleration, and Forces, page 63

The mistake of Aristotle is explained, and the law of inertia is introduced.

The concepts of mass and force are discussed.

With help of some experiments force and acceleration are measured, pages 63-66.

3.3 An Interlude of Units, page 67

Units are introduced. Units for length, time, velocity, acceleration, and mass.

The unit of force is defined by its definition:

$F = ma.$

3.4 Some Simple Examples of Solving Newton's Equation, page 69

Exercise 3.2 Newton's Second Law of Motion for a Constant Force, page 71

The constant force F_z is in the z-direction.
The equation of motion is:
$\frac{dv_z}{dt} = \frac{F_z}{m}$,
where v_z, is the velocity in the z-direction of the particle with mass m.
We integrate this equation and choose as an initial condition at $t = 0 \Longrightarrow v_z(0)$.
$\int_0^t \frac{dv_z}{dt'} dt' = \int_0^t \frac{F_z}{m} dt' \Longrightarrow v_z|_0^t = \frac{F_z}{m}|_0^t \Longrightarrow v_z(t) - v_z(0) = \frac{F_z}{m} \cdot t$,
or
$v_z(t) = v_z(0) + \frac{F_z}{m} \cdot t$.

Exercise 3.3 Differentiate the Integral of $v_z(t)$ in Exercise 3.2, page 71

Show by differentiation that:
$z(t) = z(0) + v_z(0) \cdot t + \frac{F_z}{2m} \cdot t^2$,
satisfies the equation of motion $\frac{dv_z}{dt} = \frac{F_z}{m}$.
$\frac{d}{dt}[z(t) = z(0) + v_z(0) \cdot t + \frac{F_z}{2m} \cdot t^2] \Longrightarrow v_z(t) = v_z(0) + \frac{F_z}{m} \cdot t$,
and
$\frac{d}{dt}[v_z(t) = v_z(0) + \frac{F_z}{m} \cdot t] \Longrightarrow \frac{dv_z}{dt} = \frac{F_z}{m}$.

The equation of motion of the 1-D harmonic oscillator is:
$\frac{d^2x}{dt^2} = -\omega^2 x$, Eq.(6), page 72.

Exercise 3.4 The harmonic oscillator, page 72

Show by differentiation that the general solution to $\frac{d^2x}{dt^2} = -\omega^2 x$, is given in terms of two constants A and B by
$x(t) = A\cos\omega t + B\sin\omega t$.
Determine the initial position and velocity at time $t = 0$, in terms of A and B.
$\frac{dx}{dt} = -A\omega\sin\omega t + B\omega\cos\omega t$.
Then,
$\frac{d^2x}{dt^2} = -A\omega^2\cos\omega t - B\omega^2\sin\omega t = -\omega^2(A\cos\omega t + B\sin\omega t) = -\omega^2 x$.
At $t = 0$: $x(t = 0) = A$ and $\frac{dx}{dt}(t = 0) = B\omega$.

Interlude 3: Partial Differentiation, Page 74

I.3.1 Partial Derivatives

Exercise I.3.1 Partial Derivatives

Compute all first and second partial derivatives-including mixed derivatives- of the following functions.

- $x^2 + y^2 = \sin(xy)$.

$\frac{\partial}{\partial x}: 2x = \mathrm{y}\cos(xy)\ ; \frac{\partial}{\partial y}: 2y = x\cos(xy)$.

$\frac{\partial^2}{\partial x^2}: 2 = -y^2 \sin(xy)\ ; \frac{\partial^2}{\partial y^2}: 2 = -x^2 \sin(xy)$.

$\frac{\partial^2}{\partial y \partial x}:\ 0 = \cos(xy) - xy\sin(xy);\ \frac{\partial^2}{\partial x \partial y}:\ 0 = \cos(xy) - xy\sin(xy)$.

- $\frac{x}{y} e^{x^2+y^2}$.

$\frac{\partial}{\partial x}:\ [\frac{1+2x^2}{y} e^{x^2+y^2}\ ; \frac{\partial}{\partial y}: [-\frac{x}{y^2} + 2x]e^{x^2+y^2}$.

$\frac{\partial^2}{\partial x^2}: \frac{6x+4x^3}{y} e^{x^2+y^2}\ ;\ \frac{\partial^2}{\partial y^2}: [\frac{2x}{y^3} - \frac{2x}{y} + 4xy]e^{x^2+y^2}$.

$\frac{\partial^2}{\partial y \partial x}: \left[-\frac{1+2x^2}{y^2} + 2 + 4x^2\right] e^{x^2+y^2};\ \frac{\partial^2}{\partial x \partial y}: [-\frac{1+2x^2}{y^2} + 2 + 4x^2]e^{x^2+y^2}$.

- $e^x \cos y$.

$\frac{\partial}{\partial x}: e^x \cos y\ ;\ \frac{\partial}{\partial y}: -\ e^x \sin y$.

$\frac{\partial^2}{\partial x^2}: e^x \cos y\ ;\ \frac{\partial^2}{\partial y^2}: -\ e^x \cos y$.

$\frac{\partial^2}{\partial y \partial x}:\ -e^x \sin y\ ; \frac{\partial^2}{\partial x \partial y}:\ -e^x \sin y$.

I.3.2 Stationary Points and Minimizing Functions

There are local minima and global minima, page 77.
Just below Figure.3, page 78: "*If the second derivative is equal to 0, then the derivative …..*".I prefer : "…..*then the second derivative changes*……" like the first derivative change sign where the first derivative is zero.

I.3.3 Stationary points in Higher Dimensions

On page 82, the definition of the Hessian matrix is given for a two dimensional case. Then , we have a 2×2 matrix. The determinant and the trace, the sum of the diagonal elements of the matrix, of the matrix are the numbers to determine whether you find a local minimum, of local maximum or a saddle point.
Keep in mind, this is about a 2-dimensional case.

Exercise I.3.2 About Stationary Points

Consider the set of points (x_i, y_i): $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$.

Is this set of points a stationary set of the following functions? If so, of what type?

- $F(x, y) = \sin x + \sin y$.

$\frac{\partial F}{\partial x} = \cos x$, $\frac{\partial F}{\partial y} = \cos y$;

$\frac{\partial^2 F}{\partial x^2} = -\sin x$, $\frac{\partial^2 F}{\partial y^2} = -\sin y$;

$\frac{\partial^2 F}{\partial y \partial x} = 0$, $\frac{\partial^2 F}{\partial x \partial y} = 0$.

The determinant of the Hessian matrix: $H = \begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{vmatrix} = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 F}{\partial y \partial x}$.

So,

$DetH = \sin x \sin y$.

The trace of the matrix H : $TrH = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}$.

Then,

$TrH = -\sin x - \sin y$.

What are the conclusions for the set of points (x_i, y_i)?

- $(x, y) = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$: $DetH = \sin x \sin y = -1$, and $TrH = -\sin x - \sin y = 0$.

Page 82: If the determinant is negative, then// irrespective of the trace, the point is a saddle point.

- $(x, y) = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$: $DetH = \sin x \sin y = -1$, and $TrH = -\sin x - \sin y = 0 \Longrightarrow$ a saddle point.

- $(x, y) = \left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$: $DetH = \sin x \sin y = 1$, and $TrH = -\sin x - \sin y = 2 \Longrightarrow$ If the determinant and the trace of the Hessian are positive then the point is a local minimum.

- $F(x, y) = \cos x + \cos y$

$\frac{\partial F}{\partial x} = -\sin x$, $\frac{\partial F}{\partial y} = -\sin y$;

$\frac{\partial^2 F}{\partial x^2} = -\cos x$, $\frac{\partial^2 F}{\partial y^2} = -\cos y$;

$\frac{\partial^2 F}{\partial y \partial x} = 0$, $\frac{\partial^2 F}{\partial x \partial y} = 0$.

For all the points (x_i, y_i): $\left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$; $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; $\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) \Longrightarrow \frac{\partial F}{\partial x} \neq 0$, $\frac{\partial F}{\partial y} \neq 0$.

No stationary points.

Lecture 4: Systems of More Than One Particle, Page 85

4.1 Systems of particles, page 85

"What is it that determines the force on a given particle? It is the positions of all the other particles".

Fundamental forces are, i.e., gravity and electric forces.

4.2 The space of States of a System of Particles, page 88

The meaning of the state of a system is described.

At the bottom of page 89, just below the equation of motion: "*Since there is no expression for the velocity here, let's add to this another equation expressing the fact that the velocity is the rate of change of position".* Then, at the top of page 90, the rate of change of position is given.

Eq. (2), page 90, represents the equations for N particles.

4.3 Momentum and Phase Space, page 90

Eq. (3), page 92, represents the set of equations of the phase space.

Configuration space plus momentum space equals phase space.

4.4 Action, Reaction, and Conservation of Momentum, page 92

"The principle of the conservation of momentum is a profound consequence of abstract general principles of classical mechanics that we have yet to formulate".

Top page 94:

$\sum_i \frac{d}{dt}\vec{p}_i = \sum_i \sum_j \vec{f}_{ij} = 0,$

since,

$\vec{f}_{ij} + \vec{f}_{ji} = 0 \Longrightarrow \sum_i \frac{d}{dt}\vec{p}_i = 0,$

and $\vec{f}_{ii} = 0$, since $\vec{f}_{ij} + \vec{f}_{ji} = 0 \rightarrow \vec{f}_{ii} + \vec{f}_{ii} = 0$

So, we have the mathematical formulation of the conservation of momentum;

$\sum_i \frac{d}{dt}\vec{p}_i = 0$: the total momentum of an isolated system never changes.

Lecture 5: Energy, page 95

5.1 Force and Potential Energy, page 95

Classical physics has only two forms of energy: kinetic and potential.

The sum of potential and kinetic energy is conserved.

Exercise 5.1 Kinetic Energy

Prove Eq.(3): $\frac{dv^2}{dt} = 2v\frac{dv}{dt}$.

$\frac{dv^2}{dt} = \frac{d}{dt}v \cdot v = v \cdot \frac{dv}{dt} + \frac{dv}{dt} \cdot v = 2v\frac{dv}{dt}$.

5.2 More Than One Dimension, page 99

Page 99: *"… but nature does not make use of such nonconservative forces"*.

Nonconservative means, in one dimension,

$m\frac{d^2x}{dt^2} + \frac{dV}{dx} \neq 0?$

Page 100, Eq.(5): the most important principle of physics:

$F_i(\{x\}) = -\frac{\partial V(\{x\})}{\partial x_i} \rightarrow$ the conservation of energy.

The next pages summarize the derivation of this conservation, pages 101-102

Top page 102:

basically $\frac{dV}{dt} = \frac{dV}{dx}\frac{dx}{dt}$.

Eq.(8), page 102,

$\frac{d}{dt}E = 0,$

just what was found on pages 98 and 99.

At the bottom of page 102: *"At some point we have to give up and say that's just the way it is".* Gödel for the time being of forever.

Exercise 5.2 Energy and the Equations of Motion 1

Consider a particle in two dimensions, x and y. The particle has mass m. The potential energy is $V = \frac{1}{2}k(x^2 + y^2)$. Work out the equations of motion. Show that there are circular orbits and that all the orbits have the same period. Prove that the total energy is conserved.

- Equations of motion.

$m\frac{d^2\{x\}}{dt^2} = F(\{x\})$.

$F(x) = -\frac{\partial V(\{x\})}{\partial x} = -k \cdot x$, and $F(y) = -\frac{\partial V(\{x\})}{\partial y} = -k \cdot y$.

So,

$m\frac{d^2\{x\}}{dt^2} = -k \cdot \{x\}$ or,

$m\frac{d^2x}{dt^2} = -k \cdot x$, and $m\frac{d^2y}{dt^2} = -k \cdot y$.

- Conservation of energy.

Kinetic energy: $T = \frac{1}{2}m[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2]$,

and

$\frac{dT}{dt} = m[\frac{dx}{dt} \cdot \frac{d^2x}{dt^2} + \frac{dy}{dt} \cdot \frac{d^2y}{dt^2}]$.

Furthermore

$\frac{dV}{dt} = \frac{dx}{dt} \cdot \frac{\partial V}{\partial x} + \frac{dy}{dt} \cdot \frac{\partial V}{\partial y} = -\frac{dx}{dt} \cdot m\frac{d^2x}{dt^2} - \frac{dy}{dt} \cdot m\frac{d^2y}{dt^2} = -\frac{dT}{dt}$.

Consequently,

$\frac{dV}{dt} + \frac{dT}{dt} = 0$.

- Circular orbits.

I suppose with circular orbits I can write, with r to be the radius of the circular motion,

$x = r\cos\theta(t)$,

and

$y = r\sin\theta(t)$,

where r is a constant of motion.

The potential can be written as:

$V = \frac{1}{2}k(x^2 + y^2) = \frac{1}{2}k \cdot r^2$, a central potential in two dimensions.

The force experienced by the particle:

$\vec{F} = -kx - ky = -kr[\cos\theta(t) + \sin\theta(t)]$.

So, there are circular orbits. Do all the orbits have the same period? Well, I suppose this question suggest:

$\theta = \omega t$,

where ω is a constant of motion. So, $\frac{d\theta}{dt} = \omega$, $(\frac{d\theta}{dt})^2 = \omega^2$, and $\frac{d^2\theta}{dt^2} = 0$.

I use $\frac{d^2\theta}{dt^2} = 0$.

Let's use the equation of motion: $m\frac{d^2\vec{x}}{dt^2} = \vec{F}$.

$m\frac{d^2\vec{x}}{dt^2} \Rightarrow mr\left\{\cos\theta\left[\frac{d^2\theta}{dt^2} - \left(\frac{d\theta}{dt}\right)^2\right]\right\} - \sin\theta[\frac{d^2\theta}{dt^2} + \left(\frac{d\theta}{dt}\right)^2]\} = -kr\ (\cos\theta + \sin\theta) \Rightarrow$

$\Rightarrow -m\left(\frac{d\theta}{dt}\right)^2 (\cos\theta + \sin\theta) = -k\ (\cos\theta + \sin\theta) \Rightarrow m\omega^2 = k \Rightarrow \omega = \sqrt{\frac{k}{m}}$.

Well, no surprise, I considered ω to be a constant of motion. This assumption is not contradicted.

Conservation of energy: use the analysis of page 101 and 102.

Furthermore, with

$\frac{d\vec{x}}{dt} = r\frac{d\theta}{dt}(-\sin\theta\frac{d\theta}{dt} + \cos\theta)$,

and

$\frac{d^2\vec{x}}{dt^2} = -r\left(\frac{d\theta}{dt}\right)^2 (\cos\theta + \sin\theta)$,

the total energy is conserved, Eq.(8) page 102.

Remark: I could have plugged into $m\frac{d^2x}{dt^2} = -k \cdot x$, and $m\frac{d^2y}{dt^2} = -k \cdot y$, solutions like $x = \sin\omega t$, and y= $\sin\omega t$. In this way I could have proven the existence of circular orbits.

Exercise 5.3 Energy and the Equations of Motion 2

Rework Exercise 5.2 for the potential $V = \frac{k}{2(x^2+y^2)}$. Are there circular orbits? If so, do they all have the same period? Is the total energy conserved.

- Equations of motion.

$m\frac{d^2\{x\}}{dt^2} = F(\{x\})$.

$F(x) = -\frac{\partial V(\{x\})}{\partial x} = -k \cdot \frac{x}{x^2+y^2}$, and $F(y) = -\frac{\partial V(\{x\})}{\partial y} = -k \cdot \frac{y}{x^2+y^2}$.

So,

$m\frac{d^2x}{dt^2} = -k \cdot \frac{x}{x^2+y^2}$, and $m\frac{d^2y}{dt^2} = -k \cdot \frac{y}{x^2+y^2}$.

- Are there circular orbits?

So, are

$x = \cos\omega t$, and $y = \sin\omega t$, solutions of the equations of motion?

$\frac{d^2x}{dt^2} = -\omega^2\cos\omega t$.

Plug $\frac{d^2x}{dt^2} = -\omega^2\cos\omega t$, and $x = \cos\omega t$, into $m\frac{d^2x}{dt^2} = -k \cdot \frac{x}{x^2+y^2}$:

$-m\omega^2\cos\omega t = -k \cdot \frac{\cos\omega t}{\cos^2\omega t+\sin^2\omega t} = -k\cos\omega t$.

Now,

$\frac{d^2y}{dt^2} = -\omega^2\sin\omega t$.

Plug $\frac{d^2y}{dt^2} = -\omega^2\sin\omega t$, and $y = \sin\omega t$, into $m\frac{d^2y}{dt^2} = -k \cdot \frac{y}{x^2+y^2}$:

$-m\omega^2\sin\omega t = -k \cdot \frac{\sin\omega t}{\cos^2\omega t+\sin^2\omega t} = -k\sin\omega t$.

Hence with the frequency

$\omega = \sqrt{\frac{k}{m}}$, there are circular orbits with the same frequency.

- Conservation of energy.

Kinetic energy: $T = \frac{1}{2}m[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2]$,

and

$\frac{dT}{dt} = m[\frac{dx}{dt} \cdot \frac{d^2x}{dt^2} + \frac{dy}{dt} \cdot \frac{d^2y}{dt^2}]$.

Furthermore

$\frac{dV}{dt} = \frac{dx}{dt} \cdot \frac{\partial V}{\partial x} + \frac{dy}{dt} \cdot \frac{\partial V}{\partial y} = -\frac{dx}{dt} \cdot m\frac{d^2x}{dt^2} - \frac{dy}{dt} \cdot m\frac{d^2y}{dt^2} = -\frac{dT}{dt}$.

Consequently,

$\frac{dV}{dt} + \frac{dT}{dt} = 0$.

At the end of this lecture, Susskind summarized the different kinds of energy we deal with in physics.

Lecture 6: The Principle of Least Action, Page 105

6.1 The Transition to Advanced Mechanics, page 105

"The principle of least action-really the principle of stationary action-is the most compact form of the classical laws of physics."

The problem to be solved for a moving particle is: *"Find the shortest path between the begin and endpoints $\Longrightarrow$ it is about the path of stationary action."*

6.2 Action and the Lagrangian, page 107

On page 108, the action is presented by Eq. (1).
To find the action, the Lagrangian L is needed:

$\mathcal{A} = \int_{t_0}^{t_1} L(x,\dot{x})dt$, Eq. (2) pag.109.

The Euler-Lagrange equations are needed for minimizing the action.
At the top of page 111, Susskind presented the Euler-Lagrange equation for a single degree of freedom:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

6.3 Derivation of the Euler-Lagrange Equation, page 111

The integral of the Lagrangian is replaced by a sum, page 111.
The sum being taken over the small intervals between neighbouring instants Δt.
The differentiation with respect to time is replaced by

$$\frac{dx}{dt} = \frac{x_{n+1}-x_n}{\Delta t},$$

and the position is averaged between two neighbouring points,

$$x(t) = \frac{x_n+x_{n+1}}{2}.$$

Then the total action is presented as a sum in Eq. (3), page 112:

$$L(x,\dot{x}) \Rightarrow L(\frac{x_n+x_{n+1}}{2},\frac{x_{n+1}-x_n}{\Delta t})$$

The next step is to minimize the action by varying x_n over the small intervals.
Notice, x_n is found in two intervals. Consequently, varying x_n we need to take into account those two intervals
To minimize the action, we must minimize:

$$A = L\left(\frac{x_n+x_{n+1}}{2},\frac{x_{n+1}-x_n}{\Delta t}\right) + L(\frac{x_{n-1}+x_n}{2},\frac{x_n-x_{n-1}}{\Delta t})..$$

In order to minimize the action differentiate the preceding expression with respect to x_n:
$\frac{\partial A}{\partial x_n}$ and set $\frac{\partial A}{\partial x_n} = 0$.
With $\dot{x}$, we have $\frac{\partial A}{\partial x_n} = \frac{\partial L}{\partial \dot{x}} \cdot \frac{\partial \dot{x}}{\partial x_n}$.
So, for the velocity dependence we obtain:

$$\frac{\partial}{\partial x_n} = \frac{\partial L}{\partial \dot{x}} \cdot \frac{\partial}{\partial x_n}\left[\frac{x_{n+1}-x_n}{\Delta t} + \frac{x_n-x_{n-1}}{\Delta t}\right] = -\frac{1}{\Delta t}\cdot\frac{\partial L}{\partial \dot{x}}|_{n+1} + \frac{1}{\Delta t}\cdot\frac{\partial L}{\partial \dot{x}}|_{n-1} =$$

$$= \frac{1}{\Delta t}\left(-\frac{\partial L}{\partial \dot{x}}|_{n+1} + \frac{\partial L}{\partial \dot{x}}|_{n-1}\right) \Rightarrow \lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(-\frac{\partial L}{\partial \dot{x}}|_{n+1} + \frac{\partial L}{\partial \dot{x}}|_{n-1}\right) = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}.$$

Now, the position dependency:

$$\frac{\partial A}{\partial x_n} = \frac{\partial L}{\partial x}\cdot\frac{\partial}{\partial x_n}[\frac{x_n+x_{n+1}}{2} + \frac{x_{n-1}+x_n}{2}] = \frac{1}{2}[\frac{\partial L}{\partial x}|_{n+1} + \frac{\partial L}{\partial x}|_{n-1}] \Rightarrow$$

$$\Rightarrow \lim_{(n+1)-(n-1)\to 0}\frac{1}{2}\left[\frac{\partial L}{\partial x}|_{n+1} + \frac{\partial L}{\partial x}|_{n-1}\right] = \frac{\partial L}{\partial x}.$$

Hence, the condition $\frac{\partial A}{\partial x_n} = 0$, leads us to

$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0$, Eq. (4) page 114.

Exercise 6.1 Newton, the Lagrangian and the Euler-Lagrange equation.

Show that Eq. (4) page 114,Is just another form of Newton's equation of motion $F = ma$.
The Lagrangian:
$L = T - V = \frac{1}{2}m\dot{x}^2 - V(x).$
The Euler-Lagrange equation:
$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0, Eq. (4).$
First $\frac{\partial L}{\partial \dot{x}}$
$\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right) = m\dot{x},$
Then
$m\frac{d}{dt}\dot{x} = m \cdot \ddot{x}$.
Finally, $\frac{\partial L}{\partial x}$, only V is dependent on x,
$\frac{\partial L}{\partial x} = -\frac{\partial V}{\partial x} = F.$
So,
$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \Longrightarrow m \cdot \ddot{x} - F = 0 \Longrightarrow F = m\ddot{x}$, QED.

6.4 More Particles and More Dimensions, page 114

Here the Lagrangian and the Euler-Lagrange equation for a many particle system is presented.
"*The principle of least action for more degrees of freedom is essentially no different than the case with a single degree of freedom".*
There is a Euler-Lagrange equation for each coordinate(or variable for that matter), Eq. (6) page 115.

Exercise 6.2 Euler-Lagrange and Newton for many particles

Show that $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$, is just another form of Newton's equation of motion $F_i = m\ddot{x}_i$.
The Lagrangian $L = \sum_i \frac{1}{2}m\dot{x}_i^{\,2} - V(\{x\}).$
$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i$,
and
$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = m\ddot{x}_i$.
$\frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_i.$
Hence, with
$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0 \Longrightarrow F_i = m\ddot{x}_i.$

6.5 What's Good about Least Action? Page 116

First: Least action describes a system in a concise way using a single function, the Lagrangian.
Second: Practical advantage: the Lagrangian formulation of mechanics.
Susskind illustrated this with an example: the case of a particle in one dimension with two observes moving with respect to each other. The difference in origin between the two observers is described with a function $f(t)$.

The action for the two observers is presented on page 117.
The action for the observer at rest is:
$\mathcal{A} = \int_{t_0}^{t_1} L(x, \dot{x}) dt.$
For the moving observer:
$\mathcal{A} = \int_{t_0}^{t_1} L(X + f, \dot{X} + \dot{f}) d.$
Then , with the expression for the Lagrangian : Kinetic- minus Potential energy for the moving observer, Eq. (8):
$L = \frac{1}{2} m(\dot{X} + \dot{f})^2 - V(X),$
or
$L = \frac{1}{2} m\left(\dot{X}^2 + 2\dot{X}\dot{f} + \dot{f}^2\right) - V(X).$
I gave here both equations for the Lagrangian. Since, is it:
$\frac{\partial L}{\partial \dot{X}}$, or $\frac{\partial L}{\partial(\dot{X}+\dot{f})}$?
To obtain the equation of motion on page 118:
$m\ddot{X} + m\ddot{f} = -\frac{dV}{dX},$
the first Lagrangian, $\frac{1}{2} m(\dot{X} + \dot{f})^2 - V(X)$,needs to be differentiated with respect to $\dot{X} + \dot{f}$, and the second, $\frac{1}{2} m\left(\dot{X}^2 + 2\dot{X}\dot{f} + \dot{f}^2\right) - V(X)$,with respect to $\dot{X}$. Giving in both cases:
$m(\dot{X} + \dot{f}).$
On the other hand, write $\left(\dot{X}^2 + 2\dot{X}\dot{f} + \dot{f}^2\right)$ as $[\dot{X}\left(\dot{X} + \dot{f}\right) + \dot{f}(\dot{X} + \dot{f})]$ and the second expression can be differentiated with respect to $\dot{X} + \dot{f}$, as well.
Then another example is presented. It is about a carousel rotating over the angle φ .
The relations between the frame at rest (x, y) and the moving frame (X, Y) are:
$x = X \cos\varphi + Y \sin\varphi$,
and
$y = -X \sin\varphi + Y \cos\varphi.$
The carousel rotates with a constant frequency of rotation $\omega \Longrightarrow \varphi = \omega \cdot t$.
So, Eq.(9), page 118,
$x = X \cos\omega t + Y \sin\omega t,$
$y = -X \sin\omega t + Y \cos\omega t.$
The Lagrangian for the observer at rest is given, Eq. (10), page 119.
Then, by differentiating with respect to time of the two preceding equations, Eq. (11) page 119 is found. Finally, the Lagrangian for the moving observer is obtained, Eq. (12) page 120.
It is about, kinetic energy, potential energy(central potential, sort of) and the last term leads to the Coriolis force.
The equations of motion are presented. These equations are derived in the following exercise.

Exercise 6.3 The Equations of Motion for the Carousel

Use the Euler-Lagrange equations to derive the equations of motion from the Lagrangian presented in Eq. (12) page 120:

$L = \frac{m}{2}\left(\dot{X}^2 + \dot{Y}^2\right) + \frac{m\omega^2}{2}(X^2 + Y^2) + m\omega\left(\dot{X}Y - \dot{Y}X\right)$.

Euler -Lagrange : $\frac{d}{dt}\frac{\partial L}{\partial \dot{X}_i} - \frac{\partial L}{\partial X_i} = 0$.

$\frac{\partial L}{\partial \dot{X}_i}$:

$\frac{\partial L}{\partial \dot{X}} = m\dot{X} + m\omega Y$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{X}} = m\ddot{X} + m\omega\dot{Y}$, and $\frac{\partial L}{\partial X} = m\omega^2 X - m\omega\dot{Y}$,

$\frac{\partial L}{\partial \dot{Y}} = m\dot{Y} - m\omega X$, $\frac{d}{dt}\frac{\partial L}{\partial \dot{Y}} = m\ddot{Y} - m\omega\dot{X}$, and $\frac{\partial L}{\partial Y} = m\omega^2 Y + m\omega\dot{X}$.

So both equations of motion are:

$\ddot{X} = \omega^2 X - 2\omega\dot{Y}$,

$\ddot{Y} = \omega^2 Y + 2\,\omega\dot{X}$.

Exercise 6.4 The Equations of Motion in Polar Coordinates, page 121

Work out the Lagrangian of Eq. (12) page 120 and the Euler-Lagrange equations in polar coordinates.
I think Eq.(10) page 119 should be worked out considering the remark on page 124 and Exercise 6.4.

$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$,

the Euler-Lagrange Equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$.

Hence : $m\ddot{x} = 0$ and $m\ddot{y} = 0$.

$x = r\cos\theta$,

$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$,

$\ddot{x} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta$,

$y = r\sin\theta$,

$\dot{y} = \dot{r}\sin\theta + r\,\dot{\theta}\cos\theta$,

$\ddot{y} = \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta$.

Eq. 10, the Lagrangian $L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$, in polar coordinates:

$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$.

The equations of motion:

$\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta = 0$,

and

$\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta = 0$.

The generalized momentum conjugate to r:

$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$, and $\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$.

$\frac{dp_r}{dt} = \frac{\partial L}{\partial r} = mr\dot{\theta}^2$.

The equation of motion for r:

$m\ddot{r} = mr\dot{\theta}^2 \rightarrow \ddot{r} - r\dot{\theta}^2 = 0$.

This also follows from multiplying $\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta = 0$ with $\cos\theta$ and $\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta = 0$ with $\sin\theta$, adding both resulting expressions.

The equation of motion for the angle θ follows from the conjugate momentum to θ:

$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$.

Furthermore, $\frac{dp_\theta}{dt} = \frac{\partial L}{\partial \theta} = 0$.

Hence $\frac{d}{dt}\,mr^2\dot{\theta} = 0$.

The latter expression can be obtained by multiplying $\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta = 0$, with $\sin\theta$ and $\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta = 0$ with $\cos\theta$, subtracting both resulting expressions$\rightarrow 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \rightarrow \frac{d}{dt}\,mr^2\dot{\theta} = 0$.

6.6 Generalized Coordinates and Momenta, page 121

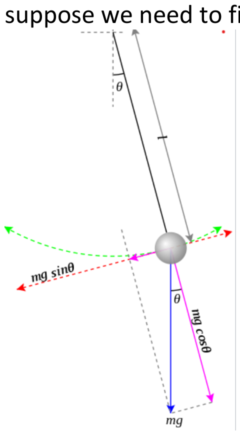
An abstract problem specified by a general set of coordinates is considered.
The generalized set of coordinates are $(q_i, \dot{q}_i)$.
Pages 122 and 123:
"*For now, we are going to take it as given that all known systems of classical physics can be described in terms of the action principle".*
"The most general form of classical equation of motion:
$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$*."* , Eq. (13).
On page 124, an example is given for a particle in polar coordinates. So,
the q_i's are the radius r and the angle θ.

Note: for this example, *the results from Exercise 6.4 can be used to get the Lagrangian*. Well, ω must be set 0 in Eq.(12). So, I would prefer to refer to Eq. (10) page 119. With use of: $x = r\cos\theta$, and $y = r\sin\theta$, the Lagrangian presented below is found. No carousel.
The Lagrangian:
$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$.
With the generalized conjugate momentum, the equations of motion are obtained. Eq. (15) page 125 demonstrates the conservation of angular momentum. That is why the angular velocity increases making a pirouette on ice while decreases r by retracting your arms.

Exercise 6.5 Pendulum and the Lagrangian, page 125

Predict the motion of a pendulum of length l.

I suppose the mass of the pendulum to be confined to the end of the rod of length l. The rod is massless. I suppose we need to find the equation of motion.



Here I showed a picture of the pendulum, Source www.en.wilipedia.org .

The Lagrangian:

L =Kinetic energy −Potential energy .

The kinetic energy: $\frac{m}{2} l^2 \dot{\theta}^2$.

So,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = ml^2 \frac{d}{dt}\dot{\theta} = ml^2 \ddot{\theta}.$$

The potential energy is at maximum for $\theta = \pi$, in the above picture.

Then,

$$V = -mgl\cos\theta.$$

$$\frac{\partial V}{\partial \theta} = mgl\sin\theta.$$

With

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial V}{\partial \theta} = 0\,,$$

the equation of motion of the pendulum is:

$$\ddot{\theta} = \frac{g}{l}\sin\theta.$$

Simple solutions can be found for small θ.

6.7 Cyclic Coordinates, page 125

Definition of cyclic coordinates: *Coordinates which do not appear in the Lagrangian are called cyclic.*

When a coordinate is cyclic, its conjugate momentum is conserved. An example of which is presented in Eq. (14) page 125.

Exercise 6.6 The Lagrangian of two particles moving on a line, page 127

The potential depends on the distance between the two particles.
The positions of the particles are x_1 and x_2.

The Lagrangian:
$L = \frac{m}{2}\left(\dot{x}_1{}^2 + \dot{x}_2{}^2\right) - V(x_1 - x_2)$.
Change the coordinates:
$x_+ = \frac{x_1+x_2}{2}$,
$x_- = \frac{x_1+x_2}{2}$.
Then
$x_1 = x_+ + x_-$,
$x_2 = x_+ - x_-$,
and
$x_1 - x_2 = 2\, x_-$.
$x_1^2 = x_+^2 + x_-^2 + 2x_+x_-$,
and
$x_2^2 = x_+^2 + x_-^2 - 2x_+x_-$.
The Lagrangian transforms with $\dot{x}_1{}^2 + \dot{x}_2{}^2 = 2(\dot{x}_+{}^2 + \dot{x}_-{}^2)$:
$L = m(\dot{x}_+{}^2 + \dot{x}_-{}^2) - V(2x_-)$.
The total momentum:
$p_+ = m(\dot{x}_1 + \dot{x}_2) = 2m\dot{x}_+$.

Lecture 7: Symmetries and Conservation Laws, Page 128

7.1 Preliminaries, page 128

The relationship between symmetries and conservation laws is one of the big main themes of modern physics.
Some examples of conservation laws are presented.
The example of two particles as presented in the preceding Lecture is analysed. The equations of motion are given in Eq. (2) page 129:
$\frac{dp_i}{dt} = \frac{\partial L}{\partial q_i}$.

Exercise 7.1 The Equations of Motion for a Two Particle System, page 129

The Lagrangian, Eq. (13) page 123:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial L}{\partial q_i}.$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i},$$

the generalized momentum conjugate of q_i .

Now, a two particle system with particles of mass $m = 1$.

The Lagrangian

$$L = \frac{1}{2}\left(\dot{q}_1{}^2 + \dot{q}_2{}^2\right) - V(q_1 - q_2).$$

Then,

$$\frac{\partial L}{\partial q_1} = -\frac{\partial V(q_1-q_2)}{\partial(q_1-q_2)}\frac{\partial(q_1-q_2)}{\partial q_1} = -\frac{\partial V(q_1-q_2)}{\partial(q_1-q_2)} \equiv -V'(q_1 - q_2)\,,$$

and

$$\frac{\partial L}{\partial q_2} = -\frac{\partial V(q_1-q_2)}{\partial(q_1-q_2)}\frac{\partial(q_1-q_2)}{\partial q_2} = \frac{\partial V(q_1-q_2)}{\partial(q_1-q_2)} \equiv V'(q_1 - q_2).$$

The equations of motion:

$$\frac{dp_i}{dt} \equiv \dot{p}_i = \frac{\partial L}{\partial q_i},$$

resulted into

$$\dot{p}_1 = -\,V'(q_1 - q_2),$$

$$\dot{p}_2 = V'(q_1 - q_2)\,.$$

Then,

$$\frac{d}{dt}(p_1 + p_2) = -V'(q_1 - q_2) + V'(q_1 - q_2) = 0.$$

Hence, momentum is conserved.

The next example. Slightly more complicated.

Exercise 7.2 The Equations of Motion for a Two Particle System, continued, page 130

A potential depended on

$$aq_1 - bq_2,$$

where a and b are numbers.

So, Eq. (3) page 129,

$$V(q_1, q_2) = V(aq_1 - bq_2).$$

Using the procedure of Exercise 7.1:

$$\dot{p}_1 = -a\,V'(aq_1 - bq_2),$$

$$\dot{p}_2 = b\,V'(aq_1 - bq_2)\,.$$

Then,

$$b\dot{p}_1 = -ba\,V'(aq_1 - bq_2),$$

$$a\dot{p}_2 = ab\,V'(aq_1 - bq_2)\,.$$

Consequently, since a and b are numbers($ab = ba$),

$$b\dot{p}_1 + a\dot{p}_2 = 0,$$

or

$$\frac{d}{dt}(bp_1 + ap_2) = 0\,.$$

Conservation is still there.

7.2 Examples of Symmetries, page 130

A symmetry is an active coordinate transformation that does not change the value of the Lagrangian.

In this Lecture translation symmetry is analysed.

The symmetry of moving a system in space by adding a constant to the coordinates is called translation symmetry.

Now, the case where the potential is dependent on $aq_1 + bq_2$, is discussed.

At the top of page 133, reference is made to Eq. (3) page 129. There we have:
$V(q_1, q_2) = V(aq_1 - bq_2)$.
Well, replace b with $-b$:
$V(q_1, q_2) = V(aq_1 + bq_2)$,
and the potential depends on $aq_1 + bq_2$. However, this is what is meant? I do not know.
Well, in the errata a revised Eq. (7) page 133 is presented:
$q_1 \to q_1 - b\delta$,
$q_2 \to q_2 + a\delta$.
I assume the Lagrangian with potential $V(aq_1 - bq_2)$ is invariant under the translation
$q_1 \to q_1 - b\delta$,
$q_2 \to q_2 + a\delta$.
In Exercise 7.3, we have the potential $V(aq_1 + bq_2)$. As shown below, Exercise 7.3, the Lagrangian with potential $V(aq_1 + bq_2)$ is invariant under the translation:
$q_1 \to q_1 + b\delta$,
$q_2 \to q_1 - a\delta$.

Exercise 7.3 Translation Symmetry and Invariance, page 133

Show that the combination of position coordinates $aq_1 + bq_2$, along with the Lagrangian are invariant under the translation.
$q_1 \to q_1 + b\delta$,
and
$q_2 \to q_2 - a\delta$,
where a and b are numbers.
-The position coordinates:
$aq_1 + bq_2 \to aq_1 + ab\delta + bq_2 - ba\delta = aq_1 + bq_2$.
-The Lagrangian:
$L = \frac{1}{2}\left(\dot{q}_1{}^2 + \dot{q}_2{}^2\right) - V(aq_1 + bq_2)$.
$\frac{1}{2}\left(\dot{q}_1{}^2 + \dot{q}_2{}^2\right) \to \frac{1}{2}\left[(\frac{d(q_1+b\delta)}{dt})^2 + \left(\frac{d(q_2-a\delta)}{dt}\right)^2\right] = \frac{1}{2}\left(\dot{q}_1{}^2 + \dot{q}_2{}^2\right)$,
$V(aq_1 + bq_2) \to V(aq_1 + ab\delta + bq_2 - ba\delta) = V(aq_1 + bq_2)$.
Hence,
both the position coordinates and the Lagrangian are invariant under the assumed translation.

Now a more complex symmetry: a particle moving in a plane under the influence of a central potential.
The Lagrangian is given in Eq. (8) page 133.
Page 134: *Since finite transformations can be compounded out of infinitesimal ones, in studying symmetries it is enough to consider transformations with small changes in the coordinates, the so-called infinitesimals transformations.*
Page 135, from the list of errata Eq. (12) should read:
the small change in the coordinate $x \to \delta x \to \delta_v x$,
the small change in the coordinate $x \to \delta y \to \delta_v y$.
Then, Eq. (12):
$\delta_v x = y\delta$,
and
$\delta_v y = -x\delta$.
Remark: I prefer δ_x, and δ_y, instead of $\delta_v x$ and $\delta_v y$. Since, it is about a small change in, e.g., coordinates and not about, e.g., $\delta_v \cdot x$.

Exercise 7.4 The Effect of Infinitesimal Transformations on the Lagrangian, page 135

Show the Lagrangian does not change to the first order in the small quantity δ.
δ is presented in the Eqs. (10)-(12) page 135. It is about rotation.
Eq. (12):
$\delta_v x = y\delta,$
and
$\delta_v y = -x\delta.$
The Lagrangian, Eq. (8) page 133,
$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2)$.
Eq. (10):
$x \to x + y\delta,$
$y \to y - x\delta.$
Eq. (11):
$\dot{x} \to \dot{x} + \dot{y}\delta,$
$\dot{y} \to \dot{y} - \dot{x}\delta.$
Plug the results of Eqs. (10) en (11) into the Lagrangian:
$L = \frac{m}{2}[(\dot{x} + \dot{y}\delta)^2 + (\dot{y} - \dot{x}\delta)^2] - V[(x + y\delta)^2 + (y - x\delta)^2] \Longrightarrow$
$\Longrightarrow L = \frac{m}{2}[(\dot{x})^2 + 2\dot{x}\dot{y}\delta + (\dot{y})^2 - 2\dot{x}\dot{y}\delta + O(\delta^2)] +$
$-V[(x)^2 + 2xy\delta + (y)^2 - 2xy\delta + O(\delta^2)] \Longrightarrow$

$\Longrightarrow L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2).$

Remark: If the potential is not a function of the distance from the origin, then the Lagrangian is not invariant with respect to the infinitesimal rotations.
An example is a potential only dependent on x. So,
$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2 + y^2) \Longrightarrow L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V(x^2)$. Then,
with an infinite rotation:
$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) - V[x^2 + 2xy\delta + O(\delta^2)] \Longrightarrow$ no invariance.

7.3 More General Symmetries, page 136

The coordinates q_i of an abstract dynamical system are considered.
The shift of coordinates is presented in Eq. (13), page 136.
At the bottom of page 136: *"If we want to know the change in velocities-in order, for example, to compute the change in the Lagrangian-we need only to differentiate Eq. (13)."*
Eq. (13):
$\delta_v q_i = f_i(q)\delta$,
differentiate this expression with respect to t,
$\frac{d}{dt}\delta_v q_i = \frac{d}{dt} f_i(q)\delta \Longrightarrow \delta_v \dot{q}_i = \frac{\partial f_i(q)}{\partial q_i}\frac{dq_i}{dt}\delta$, or
$\delta_v \dot{q}_i = \frac{\partial f_i(q)}{\partial q_i}\dot{q}_i\delta$.
From the list of errata I learned that the preceding expression $\delta_v \dot{q}_i$ is not correct.
It should be, by definition,
$\delta_v \dot{q}_i = f_i(\dot{q})\delta$,
the new Eq. (14) page 136.

Question: why the same function f for the position q and the velocity $\dot{q}$?
No explanation in the list of errata.

A symmetry for the infinitesimal case:
"A continuous symmetry is an infinitesimal transformation of the coordinates for which the change in the Lagrangian is zero."

7.4 The Consequences of Symmetry, page 137

The changes of the Lagrangian $L(q, \dot{q})$ are calculated.
In general, a small change Δ in $L(q, \dot{q})$:
$\Delta L = \frac{\partial L}{\partial \dot{q}} \Delta \dot{q} + \frac{\partial L}{\partial q} \Delta q,$
is written in the infinitesimal change $\delta_v \dot{q}_i$, and $\delta_v q_i$ of all i
$\delta_v L = \sum_i (\frac{\partial L}{\partial \dot{q}_i} \delta_v \dot{q}_i + \frac{\partial L}{\partial q_i} \delta_v q_i)$, Eq.16 page 137.
This leads to, with the Euler-Lagrange equation,
$\delta_v L = \frac{d}{dt} \sum_i p_i \, \delta_v q_i,$
with
$p_i = \frac{\partial L}{\partial \dot{q}_i}$ and $\frac{\partial L}{\partial q_i} = \frac{d p_i}{dt}$.
Now, symmetry means
$\delta_v L = 0 \therefore \frac{d}{dt} \sum_i p_i \, \delta_v q_i = 0.$
With Eq. (13) page 136, $\delta_v q_i = f_i(q)\delta,$
$\frac{d}{dt} \sum_i p_i \, \delta_v q_i = 0 \Longrightarrow \delta \frac{d}{dt} \sum_i p_i \, f_i(q) = 0 \Longrightarrow \frac{d}{dt} \sum_i p_i \, f_i(q) = 0$, Eq. (17) page 139.
"*The conservation law is proved*".
Eq. (18) page 139,
$Q = \sum_i p_i \, f_i(q),$
does not change with time, it is conserved.

7.5 Back to Examples, page 139

Eq. (18),
$Q = \sum_i p_i \, f_i(q)$
is applied to the foregoing examples.
The first example, page 128, with the potential $V(q_1 - q_2)$:
"*In this first example the variations of the coordinates in Eq. (6), page 132, defines both f_1 and f_2 to be* 1".
It is about:
$q_1 \to q_1 + \delta \Longrightarrow \delta_{q_1} = \delta \therefore f_1 = 1$, Eq. (13).
$q_2 \to q_2 + \delta \Longrightarrow \delta_{q_2} = \delta \therefore f_2 = 1$, Eq. (13).

Notice: I changed from, e.g., $\delta_v q_i \Longrightarrow \delta_{q_i}$. See remark at the bottom of page 28.

Eq. (18): $Q = p_1 + p_2 \Longrightarrow \frac{d}{dt}(p_1 + p_2) = 0$ ∴momentum is conserved.
"*For any system of particles, if the Lagrangian is invariant under simultaneously translation of the position of particles, then momentum is conserved",*
and
"*Nothing in the laws of physics changes if everything is simultaneously shifted in space".*
The second example.
It is about the variations of Eq. (7) page 133:
$q_1 \to q_1 + b\delta \Longrightarrow \delta_{q_1} = b\delta \therefore f_1 = b$, Eq. (13).
$q_2 \to q_2 - a\delta \Longrightarrow \delta_{q_2} = -a\delta \therefore f_2 = -a$, Eq. (13).

The last example is about rotation of Eq. (12) page 135:
$x \to x + y\delta \Longrightarrow \delta_x = y\delta \therefore f_1 = y$, Eq. (13).
$y \to y - x\delta \Longrightarrow \delta_x = -x\delta \therefore f_2 = -x$, Eq. (13).
Eq. (18): $Q \equiv l = p_x y - p_y x$.
Now the coordinates and momenta are conserved.
"For any system of particles, if the Lagrangian is invariant under simultaneous rotation of the position of all particles, about the origin, then angular momentum is conserved".

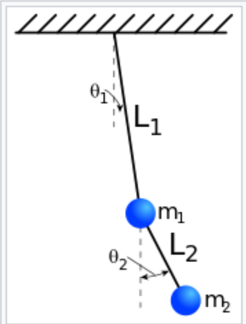
Exercise 7.5 About Conservation of Momentum, page 140

Determine the equation of motion for a simple pendulum of length r swimging through an arc in the x, y plane from an initial angle θ (see also Exercise 6.5).
The result of Exercise 6.5:
the Euler-Lagrange equation,
$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial V}{\partial \theta} = 0$,
the equation of motion of the pendulum is:
$\ddot{\theta} = \frac{g}{r}\sin\theta$.
From the Euler-Lagrange equation, we obtain:
$\frac{dp_\theta}{dt} = \frac{\partial V}{\partial \theta}$.
With
$V = -mgr\cos\theta$,
$\frac{dp_\theta}{dt} = mgr\sin\theta$.
Hence, momentum is not conserved in a gravity field.
Note: for the length of the pendulum I used r instead of l. Just above the text of Exercise 7.5 on page 140 l is used to denote angular momentum.

The double pendulum is introduced to make a convincing case for using the Lagrangian formulation.
On pages 141-143 the equations for the double pendulum are presented. At the top of page 143, the Lagrangian without gravity is presented. In addition, if there is gravity, the potential is given.

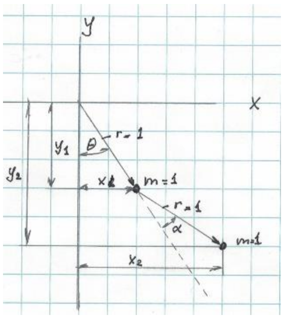
Exercise 7.6 The Lagrangian and the Double Pendulum, page 143

Work out the Euler-Lagrange equations for θ and α.
Here I showed a picture of the double pendulum, Source www.en.wilipedia.org .



A double pendulum consists of two pendulums attached end to end.

Now we set $L_1 = L_2 = 1$m, and $m_1 = m_2 = 1$ kg. Furthermore $\theta_1 = \theta$, and $\theta_2 = \theta + \alpha$.
In the figure below, the setup is given including the coordinates.



For convenience I present here the equations as derived by Susskind.

$x_1 = \sin\theta,$
$y_1 = \cos\theta,$
$x_2 = \sin\theta + \sin(\theta+\alpha),$
$y_2 = \cos\theta + \cos(\theta+\alpha)$.

The kinetic energy:

$T_1 = \frac{1}{2}(\dot{x}_1^2 + \dot{y}_1^2) = \frac{1}{2}\dot{\theta}^2(\cos^2\theta + \sin^2\theta) = \frac{1}{2}\dot{\theta}^2.$

$T_2 = \frac{1}{2}(\dot{x}_2^2 + \dot{y}_2^2) = \frac{1}{2}\left[\{\dot{\theta}\cos\theta + (\dot{\theta}+\dot{\alpha})\cos(\theta+\alpha)\}^2 + \{-\dot{\theta}\sin\theta - (\dot{\theta}+\dot{\alpha})\sin(\theta+\alpha)\}^2\right].$

Using $\cos^2\theta + \sin^2\theta = 1,$

$T_2 = \frac{1}{2}[\dot{\theta}^2 + (\dot{\theta}+\dot{\alpha})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\alpha})\{\cos\theta\cos(\theta+\alpha) + \sin\theta\sin(\theta+\alpha)\}]$.

With

$\cos\theta\cos(\theta+\alpha) + \sin\theta\sin(\theta+\alpha) = \cos(\theta-\theta-\alpha) = \cos -\alpha = \cos\alpha$,

$T_2 = \frac{1}{2}\left[\dot{\theta}^2 + (\dot{\theta}+\dot{\alpha})^2\right] + \dot{\theta}(\dot{\theta}+\dot{\alpha})\cos\alpha$.

I include potential energy

$V = -g[2\cos\theta + \cos(\theta+\alpha)].$

All the ingredients are available

- $\frac{\partial L}{\partial\dot{\theta}} = 3\dot{\theta} + \dot{\alpha} + (2\dot{\theta}+\dot{\alpha})\cos\alpha \Longrightarrow \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 3\ddot{\theta} + \ddot{\alpha} + (2\ddot{\theta}+\ddot{\alpha})\cos\alpha - (2\dot{\theta}+\dot{\alpha})\dot{\alpha}\sin\alpha,$

Note: here is the reason why I do not like the fluxion $\frac{\partial L}{\partial\dot{\theta}} = \frac{\partial L}{\partial\frac{d\theta}{dt}}$. The fluxion is difficult to discern.

- $\frac{\partial L}{\partial\dot{\alpha}} = \dot{\theta} + \dot{\alpha} + \dot{\theta}\cos\alpha, \Longrightarrow \frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}} = \ddot{\theta} + \ddot{\alpha} + \ddot{\theta}\cos\alpha - \dot{\theta}\dot{\alpha}\sin\alpha,$
- $\frac{\partial L}{\partial\theta} = 2g\sin\theta + g\sin(\theta+\alpha),$
- $\frac{\partial L}{\partial\alpha} = g\sin(\theta+\alpha).$

The Euler-Lagrange equation for θ:

$3\ddot{\theta} + \ddot{\alpha} + (2\ddot{\theta}+\ddot{\alpha})\cos\alpha - (2\dot{\theta}+\dot{\alpha})\dot{\alpha}\sin\alpha = 2g\sin\theta + g\sin(\theta+\alpha).$

The Euler-Lagrange equation for α:
$\ddot{\theta} + \ddot{\alpha} + \ddot{\theta}\cos\alpha - \dot{\theta}\dot{\alpha}\sin\alpha = g\sin(\theta + \alpha)$.
The conjugate momenta:
- $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = 3\dot{\theta} + \dot{\alpha} + (2\dot{\theta} + \dot{\alpha})\cos\alpha$,
- $p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = \dot{\theta} + \dot{\alpha} + \dot{\theta}\cos\alpha$.

Since in Exercise 7.6, $\frac{\partial L}{\partial \theta} \neq 0$ and $\frac{\partial L}{\partial \alpha} \neq 0$, momenta are not conserved. Without a gravitational field: $\frac{\partial L}{\partial \theta} = 0$ and $\frac{\partial L}{\partial \alpha} = 0$.

Exercise 7.7 Conservation of Angular Momentum and the Double Pendulum, page 144

Work out the form of the angular momentum for the double pendulum and prove that the angular momentum is conserved when there is no gravitational field.
All the work has been done in Exercise 7.6.
- $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = 3\dot{\theta} + \dot{\alpha} + (2\dot{\theta} + \dot{\alpha})\cos\alpha$,
- $p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = \dot{\theta} + \dot{\alpha} + \dot{\theta}\cos\alpha$.

Without the gravitational field:
$\frac{\partial L}{\partial \theta} = 0$ and $\frac{\partial L}{\partial \alpha} = 0$,
in the Euler-Lagrange equations.
Consequently,
$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0 \Longrightarrow \frac{d}{dt}p_\theta = 0$,
and
$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = 0 \Longrightarrow \frac{d}{dt}p_\alpha = 0$.
The angular momentum is conserved.

Lecture 8: Hamiltonian Mechanics and Time-Translation Invariance, Page 145

8.1 Time-Translation Symmetry, page 145

"*Energy conservation: the symmetry connected with energy conservation involves a shift of time*".
A system is time-translation invariant if there is no explicit time dependence in its Lagrangian.

8.2 Energy Conservation, page 147

In this Lecture the changes of the Lagrangian of a system are analysed when the system evolves.
So, the relevant expression to be evaluated is, Eq. (2) page 148 the Lagrangian explicit time dependency,
$\frac{dL}{dt} = \sum_i (\frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{q}_i}{\partial t}) + \frac{\partial L}{\partial t}$.
With
$\frac{\partial L}{\partial q_i} = \frac{\partial p_i}{\partial t}$, $\frac{dL}{dt}$ can be written as
$\frac{dL}{dt} = \frac{d}{dt}\sum_i p_i\,\dot{q}_i + \frac{\partial L}{\partial t}$, Eq. (3) page 149,

or
$\frac{d}{dt}(\sum_i p_i \dot{q}_i - L) = -\frac{\partial L}{\partial t}$.
When $\frac{\partial L}{\partial t} = 0$, $\sum_i p_i \dot{q}_i - L$ is conserved.
$\sum_i p_i \dot{q}_i - L$ is the Hamiltonian.
"If a system is time-translational invariant, the Hamiltonian is conserved."
Susskind illustrated the elegance of the Lagrangian and the Hamiltonian for a particle moving in a potential, page 150.
With $L = T - V$, and $H = T + V$,
$p\dot{q} - L = H \Rightarrow H = p\dot{q} - T + V \Rightarrow T + V = p\dot{q} - T + V \Rightarrow T = \frac{1}{2}p\dot{q}$,
as it should be.
Next, page 151, an example is evaluated where the Lagrangian is explicitly time-dependent.
It is about a charged particle moving between the plates of a capacitor.
The Lagrangian is:
$L = \frac{m}{2}\dot{x}^2 - \boldsymbol{\epsilon} x$,
where $\boldsymbol{\epsilon}$ is the uniform electric field due to the charges on the plate.
Now the capacitor is charged up and the field is a function of time.
At the bottom of page 151,
"…. The energy varies according to $\frac{dH}{dt} = x\frac{d\epsilon}{dt}$*."*
On page 152, to evaluate this variation with time, the total system needs to be evaluated.

Remark:
About $\frac{dH}{dt} = x\frac{d\boldsymbol{\epsilon}}{dt}$.
We have
$L = T - V$, and $T = H - V \Rightarrow L = H - 2V$.
$\frac{dL}{dt} = \frac{dH}{dt} - 2\frac{dV}{dt} = \frac{dH}{dt} - 2x\frac{d\boldsymbol{\epsilon}}{dt} - 2\boldsymbol{\epsilon}\frac{dx}{dt}$.
With $L = \frac{m}{2}\dot{x}^2 - \boldsymbol{\epsilon} x$,
$\frac{dL}{dt} = m\ddot{x} - x\frac{d\boldsymbol{\epsilon}}{dt} - \boldsymbol{\epsilon}\frac{dx}{dt}$.
Then,
$\frac{dH}{dt} = m\ddot{x} + x\frac{d\boldsymbol{\epsilon}}{dt} + \boldsymbol{\epsilon}\frac{dx}{dt}$?

8.3 Phase Space and Hamiltonian's Equations, page 152

More attention will be paid to the Hamiltonian.
Susskind started with a particle on a line.
$H = \frac{m\dot{x}^2}{2} + V(x)$, Eq.(8) page 153.
$p = m\dot{x} \Rightarrow \dot{p} = m\ddot{x} = F$.
$\dot{p} = F = -\frac{dV}{dx} = -\frac{\partial H}{\partial x}$, Eq. (11) page 155.
So,
$H = \frac{p^2}{2m} + V(x)$,
and
$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$.

8.4 The Harmonic Oscillator Hamiltonian, page 156

Exercise 8.1 The Lagrangian and the Harmonic Oscillator, page 157

Start with the Lagrangian: $\frac{m\dot{x}^2}{2} - \frac{k}{2}x^2$, and show that if you make the change in variables $q = (km)^{1/4}x$, the Lagrangian has the form of $L = \frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2$, Eq. (14), page 157.
What is the connection among k, m, and ω?
Looking at Eq. (14) page 157, one may wonder what the dimensions of q and $\dot{q}$ are.
Let us have a look:
$q = (km)^{1/4}x \Longrightarrow [k]$ is Newton/meter, $[m]$ is kg, [Newton] are kg/(meters·sec^2) and $[x]$ is meter.
So, [q] are (kg/sec)$^{1/2}$ ·meter.
Plugging these results into Eq. (14) you will find L to have the dimension of energy.
Caveat: do not assume q to be a space coordinate and $\dot{q}$ to be a velocity!
We have
$\dot{q} = (km)^{1/4}\dot{x}$.
Then, with $q = (km)^{1/4}x$, $\frac{m\dot{x}^2}{2} - \frac{k}{2}x^2$ becomes
$\frac{m}{2}\frac{\dot{q}^2}{\sqrt{km}} - \frac{k}{2}\frac{q^2}{\sqrt{km}} \Longrightarrow \frac{1}{2}\sqrt{\frac{m}{k}}\dot{q}^2 - \frac{1}{2}\sqrt{\frac{k}{m}}q^2$.
Plug into the preceding expression $\sqrt{\frac{k}{m}} = \omega$, and Eq. (14) is obtained
$L = \frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2$.

Exercise 8.2 The Hamiltonian of the Harmonic Oscillator, page 157

Use $L = \frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2$, Eq. (14) page 157, to calculate the Hamiltonian in terms of p and q.
$p = \frac{\partial L}{\partial \dot{q}} = \frac{\dot{q}}{\omega} \Longrightarrow \dot{q} = \omega p$.
Plug $\dot{q} = \omega p$ into the Lagrangian:
$L = \frac{\omega}{2}p^2 - \frac{\omega}{2}q^2$.
Then , the Hamiltonian
$H = \frac{\omega}{2}(p^2 + q^2)$.

The Hamiltonian in Eq.(15) is about a degree of freedom q.
So, I think Eqs. (16) page 158 should read:
$\dot{p}_i = -\omega q_i$,
and
$\dot{q}_i = \omega p_i$.
Now, let us derive the equation(s) of motion for the degree of freedom q and the harmonic oscillator.
Euler-Lagrange :
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0.$$
With $L = \frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2$,

$\frac{\partial L}{\partial \dot{q}} = \frac{\partial}{\partial \dot{q}}(\frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2) = \frac{\dot{q}}{\omega}$.

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\ddot{q}}{\omega} = \frac{\partial L}{\partial q} = \frac{\partial}{\partial q}\left(\frac{1}{2\omega}\dot{q}^2 - \frac{\omega}{2}q^2\right) = -\omega q.$

Hence

$\frac{\ddot{q}}{\omega} = -\omega q$, Eq. (17) page 158.

This result can be directly obtained from Eq. (16) page 158 as shown on page 158.

On page 159 the two-dimensional phase space of the harmonic oscillator is evaluated and illustrated in Fig. 1.

8.5 Derivation of Hamiltonian's Equations, page 160

I will present here, page 161, some of the errata of Hrabovsky:

Eq. (18) $\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial \dot{q}_i}$, should read $\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}$,

at the middle of page 161

$\frac{\partial L}{\partial \dot{q}_i} = \dot{p}_i$, should read $\frac{\partial L}{\partial q_i} = \dot{p}_i$.

Lecture 9: The Phase Space Fluid and the Gibbs-Liouville Theorem, Page 162

9.1 The Phase Space Fluid, page 162

It is about the moving of phase points in the phase space. The general case is evaluated, page 163.

On page 164, the harmonic oscillator phase space is presented, Figure 1, and the energy surfaces, Eq. (3),

$\frac{\omega}{2}(q^2 + p^2) = E.$

9.2 A Quick reminder, page 164

The reminder is about incoming and outgoing arrows as discussed in Lecture 1.

The expressions convergence and divergence are introduced in relation with irreversibility.

9.3 Flow and Divergence, page 165

Some simple examples of fluid flow in ordinary space are considered leading to the divergence of the velocity field, Eq. (4) page 167.

9.4 Liouville's Theorem, page

Eqs. (6) page 169⟹Eqs. (19) page 161.

Page 169: "*In classical mechanics(physics, Nz), the incompressibility of the phase space fluid is called Liouville's Theorem*".

Page 170, a simple Hamiltonian is used to illustrate the theorem.

$H = pq.$

The equations of motion, Eqs. (6) page 169,

$\frac{\partial H}{\partial p} = \dot{q},$

and

$\frac{\partial H}{\partial q} = -\dot{p}.$

Then with $H = pq$:

$\frac{\partial H}{\partial p} = q,$

and

$\frac{\partial H}{\partial q} = p.$
Combining the above expressions:
$\dot{q} = q \Longrightarrow q \propto e^{t}$
and
$-\dot{p} = p \Longrightarrow p \propto e^{-t}.$
Conclusion: in phase space the volume(blob) does not change ⟹incompressibility.

9.5 Poisson Brackets, page 171

Poisson brackets, PB: *"… a formulation of mechanics that seems to have been very prescient".*
Page 171: Consider the function $F(p,q)$.
Then, The Bracket PB, Eq. (9) page 172
$\{F,G\} = \sum_i(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i})$.
As an example, with the Hamiltonian H,
$\{q_k,H\} = \sum_i(\frac{\partial q_k}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial q_k}{\partial p_i}\frac{\partial H}{\partial q_i})$,
and only $q_i = q_k$, contributes.
So,
$\{q_k,H\} = \frac{\partial H}{\partial p_k} = \dot{q}_k$, Eqs. (19) page 161.
$\{p_k,H\} = \sum_i(\frac{\partial p_k}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial p_k}{\partial p_i}\frac{\partial H}{\partial q_i}) = -\frac{\partial H}{\partial q_k} = \dot{p}_k$, Eqs. (19) page 161.

Lecture 10: Poisson Brackets, Angular Momentum, and Symmetries, Page 174

10.1 An Axiomatic Formulation of Mechanics, page 174

A set of rules is presented to manipulate Poisson Brackets (PB's).
The definition of PB's, Eq.(1) page174,
$\{A,C\} = \sum_i(\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial C}{\partial q_i})$.

- Eq. (2) page 174, antisymmetry
$\{C,A\} = \sum_i(\frac{\partial C}{\partial q_i}\frac{\partial A}{\partial p_i} - \frac{\partial C}{\partial p_i}\frac{\partial A}{\partial q_i}) = \sum_i(-\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} + \frac{\partial A}{\partial p_i}\frac{\partial C}{\partial q_i}) = -\{A,C\}$.
Then, with $\{C,A\} = -\{A,C\}$ and $C = A$
$\{A,A\} = -\{A,A\} \Longrightarrow \{A,A\} = 0$, Eq.(3) page 174.

- Eq. (4) page 175, linearity
multiply A by a constant k (Susskind: *"but not C"*, meaning?)
$\{kA,C\} = \sum_i(\frac{\partial kA}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial kA}{\partial p_i}\frac{\partial C}{\partial q_i}) = \sum_i(k\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} - k\frac{\partial A}{\partial p_i}\frac{\partial C}{\partial q_i}) = k\{A,C\}.$
Now,
$\{A,kC\} = \sum_i(\frac{\partial A}{\partial q_i}\frac{\partial kC}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial kC}{\partial q_i}) = \sum_i(k\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} - k\frac{\partial A}{\partial p_i}\frac{\partial C}{\partial q_i}) = k\{A,C\}.$
Hence, a remark
$\{kA,C\} = \{A,kC\} = k\{A,C\}$, is that a problem?
Eq. (5) page 175, linearity
$\{(A+B),C\} = \sum_i[\frac{\partial(A+B)}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial(A+B)}{\partial p_i}\frac{\partial C}{\partial q_i}] = \sum_i(\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} + \frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial C}{\partial q_i} - \frac{\partial B}{\partial p_i}\frac{\partial C}{\partial q_i}) =$

$= \{A, C\} + \{B, C\}$.

-Eq. (6) page 175, using the product rule

$\{AB, C\} = \sum_i [\frac{\partial(AB)}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial(AB)}{\partial p_i}\frac{\partial C}{\partial q_i}] = \sum_i (A\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_i} + B\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} - A\frac{\partial B}{\partial p_i}\frac{\partial C}{\partial q_i} - B\frac{\partial A}{\partial p_i}\frac{\partial C}{\partial q_i}) =$
$= A\{B, C\} + B\{A, C\}$.

- Eqs. (7) and (8) page 175

$\{q_i, q_j\} = \sum_k (\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial q_k}) = 1 \cdot 0 - 0 \cdot 0 = 0$, for $k = i$,

and

$\{q_i, q_j\} = \sum_k (\frac{\partial q_i}{\partial q_k}\frac{\partial q_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial q_j}{\partial k}) = 0 \cdot 0 - 0 \cdot 1 = 0$, for $k = j$.

Similarly

$\{p_i, p_j\} = 0$.

Next, Eq. (8) page 175

$\{q_i, p_j\} = \sum_k (\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}) = 1$, for $k = i = j$,

and

$\{q_i, p_j\} = \sum_k (\frac{\partial q_i}{\partial q_k}\frac{\partial p_j}{\partial p_k} - \frac{\partial q_i}{\partial p_k}\frac{\partial p_j}{\partial q_k}) = 0$, for $k \neq i, j$.

Consequently

$\{q_i, p_j\} = \delta_{ij}$.

All the ingredients are there to calculate PB's.

through the analysis of Eq. (5) page 176 , the expression for any function of q and p has been derived , Eq. (14) page 177.

Exercise 10.1 A Poisson Bracket, page 177

Prove Eq. (14), page 177 :

Eq. (14) page 177 : $\{F(q, p), p_i\} = \frac{\partial F(q,p)}{\partial q_i}$.

$\{F(q, p), p_i\} = \sum_k [\frac{\partial F(q,p)}{\partial q_k}\frac{\partial p_i}{\partial p_k} - \frac{\partial F(q,p)}{\partial p_k}\frac{\partial p_i}{\partial q_k}] = \frac{\partial F(q,p)}{\partial q_k}\delta_{ki}$.

So, Eq. (15) page 178

$\{F(q, p), q_i\} = -\frac{\partial F(q,p)}{\partial p_k}\delta_{ki}$.

Exercise 10.2 PB's axioms and Newton's equations of motion, page 178

Hamilton's equations can be written in the form $\dot{q} = \{q, H\}$, and $\dot{p} = \{p, H\}$. Assume that the Hamiltonian has the form $H = \frac{1}{2m}p^2 + V(q)$. Using only PB's axioms, prove Newton's equations of motion.

$$\{q, H\} = \frac{\partial q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial q}{\partial p}\frac{\partial H}{\partial q} = \frac{\partial H}{\partial p} = \dot{q},$$

and

$$\{p, H\} = \frac{\partial p}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial p}{\partial p}\frac{\partial H}{\partial q} = -\frac{\partial H}{\partial q} = \dot{p}.$$

With $H = \frac{1}{2m}p^2 + V(q)$

$$\{q, H\} = \left\{q, \frac{1}{2m}\frac{\partial p^2}{\partial p}\right\} = \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{q}\ ,$$

and

$$\{p, H\} = \{p, V(q)\} = -\frac{\partial H}{\partial q} = -\frac{dV}{dq} = \dot{p} = F.$$

Now

$\frac{p}{m} = \dot{q} \Longrightarrow p = m\dot{q} \Longrightarrow \dot{p} = m\ddot{q}$: Newton's equation of motion.

10.2 Angular Momentum, page 178

The relation between rotation symmetry and the conservation of angular momentum is briefly reviewed, see Lecture 7 page 140. I recapitulate some results of the pages 135 and 140.

Page 135: $\delta x = y\delta$, and $\delta y = -\delta x$.

From the list of errata these expressions reads: $\delta_v x = y\delta$, and $\delta_v y = -y\delta$.

As mentioned, I consider the latter notation still confusing. I prefer:

$\delta_x = y\delta$, and $\delta_y = -y\delta$.

Page 140: $\delta_x = f_x\delta$, and $\delta_y = f_y\delta$.

Page 178: $\delta_x = \epsilon f_x = -\epsilon y$, and $\delta_y = \epsilon f_y = \epsilon x$.

Hence, in this Lecture on Angular Momentum, there is a change of sign: $\delta = -\epsilon$ **.** Why? At the bottom of page 178 a change of sign is mentioned. I suppose it is the above-mentioned change of sign. Furthermore, L , page 179, is used instead of l , page 140.

Exercise 10.3 Angular Momentum and PB's, page 180

Using both the definitions of PB's and the axioms, work out the PB's in Eq. (19) page 180.
Hint: In each expression, look for things in the parentheses that have nonzero PB's with coordinate x, y, or z. For example, in the first PB, x has a nonzero PB with p_x.
The angular momentum

$L_z = xp_y - yp_x.$

$\{q_k, L_z\} = \sum_{i=1}^{3}(\frac{\partial q_k}{\partial q_i}\frac{\partial L_z}{\partial p_i} - \frac{\partial q_k}{\partial p_i}\frac{\partial L_z}{\partial q_i})$, and $q_k = x, y, z.$

- $\{x, L_z\} = \sum_{i=1}^{3}(\frac{\partial x}{\partial q_i}\frac{\partial L_z}{\partial p_i} - \frac{\partial x}{\partial p_i}\frac{\partial L_z}{\partial q_i}),$

the only term contributing to the PB is: $\frac{\partial x}{\partial x}\frac{\partial L_z}{\partial p_x} = -y.$

So,

$\{x, L_z\} = -y.$

- $\{y, L_z\} = \sum_{i=1}^{3}(\frac{\partial y}{\partial q_i}\frac{\partial L_z}{\partial p_i} - \frac{\partial y}{\partial p_i}\frac{\partial L_z}{\partial q_i}),$

the only term contributing to the PB is: $\frac{\partial y}{\partial y}\frac{\partial L_z}{\partial p_y} = x.$

So,

$\{y, L_z\} = x.$

- $\{z, L_z\} = \sum_{i=1}^{3}(\frac{\partial z}{\partial q_i}\frac{\partial L_z}{\partial p_i} - \frac{\partial z}{\partial p_i}\frac{\partial L_z}{\partial q_i}),$

no term is contributing to the PB.

So,

$\{z, L_z\} = 0\,.$

With an eye on the above exercise, the other PB's are:

$\{x, L_x\} = \sum_{i=1}^{3}(\frac{\partial x}{\partial q_i}\frac{\partial L_x}{\partial p_i} - \frac{\partial x}{\partial p_i}\frac{\partial L_x}{\partial q_i}), \{x, L_x\} = 0\,,$

$\{y, L_x\} = \sum_{i=1}^{3}(\frac{\partial y}{\partial q_i}\frac{\partial L_x}{\partial p_i} - \frac{\partial y}{\partial p_i}\frac{\partial L_x}{\partial q_i}), \{y, L_x\} = -z,$

$\{z, L_x\} = \sum_{i=1}^{3}(\frac{\partial z}{\partial q_i}\frac{\partial L_x}{\partial p_i} - \frac{\partial z}{\partial p_i}\frac{\partial L_x}{\partial q_i}), \{z, L_x\} = y\,.$

Then,

$\{x, L_y\} = \sum_{i=1}^{3}(\frac{\partial x}{\partial q_i}\frac{\partial L_y}{\partial p_i} - \frac{\partial x}{\partial p_i}\frac{\partial L_y}{\partial q_i}), \{x, L_y\} = z,$

$\{y, L_y\} = \sum_{i=1}^{3}(\frac{\partial y}{\partial q_i}\frac{\partial L_y}{\partial p_i} - \frac{\partial y}{\partial p_i}\frac{\partial L_y}{\partial q_i}), \{y, L_y\} = 0$

$\{z, L_y\} = \sum_{i=1}^{3}(\frac{\partial z}{\partial q_i}\frac{\partial L_y}{\partial p_i} - \frac{\partial z}{\partial p_i}\frac{\partial L_y}{\partial q_i}), \{z, L_y\} = -x.$

where use has been made of the angular momenta on page 179:

$L_x = yp_z - zp_y\,,$

and

$L_y = zp_x - xp_z.$

10.3 Mathematical Interlude-The Levi-Civita Symbol, page 181

In this section a concise method is given to summarize the PB's of the foregoing Lecture. This is done by using the so-called Levi-Civita Symbol ϵ_{ijk}.
The use of the symbol is explained and illustrated in Figure 1 page 182: a circular arrangement of the numbers 1, 2, and 3.

10.4 Back to Angular Momentum, page 182

With the Levi-Civita Symbol ϵ_{ijk} the PB's can be summarized as follows:
$\{x_i, L_j\} = \epsilon_{ijk}x_k$, Eq. (20) page 182 without summation(See **Remark** below).
Susskind illustrated the efficient way to use the symbol for

$\{x_2, L_1\} \equiv \{y, L_x\} = \epsilon_{213} x_3 \equiv -z$,
as obtained in Lecture 10.2.
The symbol can be used for other PB's :
$\{p_i, L_j\} = \epsilon_{ijk} p_k$.
Remark:
In the list of errata it is mentioned:
$\{p_i, L_j\} = \epsilon_{ijk} p_k$ should read $\sum_k \epsilon_{ijk} p_k$.
Well, my question is: summation of what? $\epsilon_{ijk} \neq 0$, for $k \neq i$, $k \neq j$, and $i \neq j$.
Conclusion: there is just one value of k.
I suppose with $\{p_i, L_j\}$ the positions of i, j are fixed. Equal to the example above:
$\{x_2, L_1\} = \epsilon_{213} x_3$. If not,
$\{x_2, L_1\} = \sum_k \epsilon_{ijk} x_k = \epsilon_{123} x_3 + \epsilon_{132} x_3 + \epsilon_{213} x_3 + \epsilon_{231} x_3 + \epsilon_{312} x_3 + \epsilon_{321} x_3 = 0$.
As it should be.

Conclusion of this Lecture: The angular momentum is the generator of rotations.

10.5 Rotors and Precession, page 183

The PB's of components of $\vec{L}$, the angular momentum vector
$\vec{L} = (L_x, L_y, L_z)$,
are presented in Eq. (22) page 184. Here again, the summation is not necessary.
At the top of page 184, Susskind presented the PB
$\{L_x, L_y\} = L_z$.
This result can be obtained by using the definitions of PB's or by using the axioms. I tried both and preferred the definition:
$\{L_x, L_y\} = \sum_i \left(\frac{\partial L_x}{\partial q_i}\frac{\partial L_y}{\partial p_i} - \frac{\partial L_x}{\partial p_i}\frac{\partial L_y}{\partial q_i}\right)$.
Using
$L_x = y p_z - z p_y$,
and
$L_y = z p_x - x p_z$,
the result for
$q_1 = x \Longrightarrow \frac{\partial L_x}{\partial x}\frac{\partial L_y}{\partial p_x} - \frac{\partial L_x}{\partial p_x}\frac{\partial L_y}{\partial x} = 0$,
$q_2 = y \Longrightarrow \frac{\partial L_x}{\partial y}\frac{\partial L_y}{\partial p_y} - \frac{\partial L_x}{\partial p_y}\frac{\partial L_y}{\partial y} = 0$,
and
$q_3 = z \Longrightarrow \frac{\partial L_x}{\partial z}\frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z}\frac{\partial L_y}{\partial z} = x p_y - y p_x$.
Hence
$\{L_x, L_y\} = x p_y - y p_x = L_z$.
Next a rotor is evaluated without and with an electric charge.
At the bottom of page 184: "*In that case, there will be some energy associated with any misalignment between the angular momentum $\vec{L}$ and the magnetic field $\vec{B}$ (see Figure 2 page 185*".
Remark:
Misalignment, what does that mean? The energy associated with misalignment is at maximum when the angle between the two vectors $\vec{L}$ and $\vec{B}$ is zero: the cosine of the angle

is maximum. Is the misalignment between $\vec{L}$ and $\vec{B}$ at maximum when $\vec{L}$ and $\vec{B}$ are aligned?

To take the analysis further, page 186, the PB's formulation is used to work out the equations of motion for the vector $\vec{L}$.
"*First recall……*":
page 173,
$\dot{q}_k = \{q_k, H\}$,
$\dot{p}_k = \{p_k, H\}$.
Apply this to the components of the angular momentum vector $\vec{L}$, and use Eq. (24) page 185, the time derivatives of the angular momentum are obtained (bottom of page 186).
Now, we can apply Eq. (22) page 184:
$\{L_i, L_j\} = \boldsymbol{\epsilon}_{ijk} L_k$, without summation.
Then<
$\dot{L}_z = \omega\{L_z, L_z\} = 0,$
$\dot{L}_x = \omega\{L_x, L_z\} = \omega\boldsymbol{\epsilon}_{132} L_y = -\omega L_y.$
$\dot{L}_y = \omega\{L_y, L_z\} = \omega\boldsymbol{\epsilon}_{231} L_x = \omega L_x.$

10.6 Symmetry and Conservation, page 187

Eq. (21) page 183: $\delta F = \{F, L_i\}$.
In this Lecture a couple of examples represented by Eq. (21) page 183, are dealt with.

Lecture 11: Electric and Magnetic Forces, Page 190

11.1 Vector Fields, page 190

Examples of scalar fields and vector fields are presented.

11.2 Mathematical Interlude: Del, page 191

The components of the vector $\vec{\nabla}$ are presented, Eq. (1) page 191.
$\vec{\nabla}$ can considered to be an operator⟹ it must act on something, *derivatives of what*?, top page 192.
$\vec{\nabla}$ operating on a scalar creates a vector field.
Then , page 192 the divergence of a vector field is derived and creates a scalar.
At the top of page 193 the cross product is defined. Concise notation of the cross product is given in Eq.(3), page 193.

Exercise 11.1 About the Cross product, page 193

Confirm Eq. (3) page 193:

$(\vec{V}\times\vec{A})_i = \sum_j \sum_k \epsilon_{ijk} V_j A_k$.

$(\vec{V}\times\vec{A})_1 = \epsilon_{123} V_2 A_3 + \epsilon_{132} V_3 A_2 = V_2 A_3 - V_3 A_2,$

$(\vec{V}\times\vec{A})_2 = \epsilon_{231} V_3 A_1 + \epsilon_{213} V_1 A_3 = V_3 A_1 - V_1 A_3,$

$(\vec{V}\times\vec{A})_3 = \epsilon_{312} V_1 A_2 + \epsilon_{321} V_1 A_3 = V_1 A_2 - V_2 A_1.$

Prove:

$V_i A_j - V_j A_i = \sum_k \epsilon_{ijk} (\vec{V}\times\vec{A})_i.$

Eq. (3) page 193:

$(\vec{V}\times\vec{A})_i = \sum_j \sum_k \epsilon_{ijk} V_j A_k,$

with i fixed, there are two possibilities : a permutation of j and k.

$(\vec{V}\times\vec{A})_i = \epsilon_{ijk} V_j A_k + \epsilon_{ikj} V_k A_j$, jk is clockwise and kj counter clockwise.

Then,

$(\vec{V}\times\vec{A})_i = V_j A_k - V_k A_j.$

Now,

$\sum_k \epsilon_{ijk} (\vec{V}\times\vec{A})_i = \sum_k \epsilon_{ijk} (V_j A_k - V_k A_j),$

and $k \neq j$.

Hence, i takes the position of k in the right hand side of the preceding expression.

So, counter clockwise

$\sum_k \epsilon_{ijk} (\vec{V}\times\vec{A})_i = -(V_j A_i - V_i A_j) = V_i A_j - V_j A_i.$

On page 194 the theorem

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0,$$

is presented.

Also

$$\vec{\nabla}\times[\vec{\nabla}\,V(x)] = 0\,,$$

where $V(x)$ is the scalar field.

Exercise 11.2 Some Curls, page 194

Prove

$\vec{\nabla}\times[\vec{\nabla}\,V(x)] = 0$, Eq. (4) page 194.

$(\nabla_x, \nabla_y, \nabla_z)\times(\nabla_x V, \nabla_y V, \nabla_z V).$

Let us denote this preceding vector $\vec{D} = (D_x, D_y, D_{z)}$ and use Eq. (1) page 191,

Then, with Eq. (3) page 193,

$D_x = \nabla_y\nabla_z V - \nabla_z\nabla_y V = 0,$

$D_y = \nabla_z\nabla_x V - \nabla_x\nabla_z V = 0,$

$D_z = \nabla_x\nabla_y V - \nabla_y\nabla_x V = 0.$

I also include the proof of the other curl:

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0.$

Two ways:

- Denote $(\vec{\nabla}\times\vec{A}) \Rightarrow \vec{W}$.

$\vec{W}\perp\vec{\nabla}$, and $\vec{W}\perp\vec{\mathrm{A}}$.

Hence,

$\vec{\nabla}\cdot\vec{W} = 0.$

- Brute Force (Susskind (2)).

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = (\nabla_x, \nabla_y, \nabla_z)\cdot\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) =$

$= \frac{\partial^2 A_z}{\partial x\partial y} - \frac{\partial^2 A_z}{\partial x\partial z} + \frac{\partial^2 A_x}{\partial y\partial z} - \frac{\partial^2 A_z}{\partial y\partial x} + \frac{\partial^2 A_y}{\partial z\partial x} - \frac{\partial^2 A_x}{\partial z\partial y} = 0\,.$

11.3 Magnetic Fields, page 194

All magnetic fields have one characteristic feature: Their divergence is zero.
Adding the gradient of a scalar to the vector potential does not change the magnetic field.
Top of page 196:
$\vec{A}' = \vec{A} + \vec{\nabla}s,$
where s is a scalar.
Then, Eq. (4) page 194,
$\vec{\nabla} \times [\vec{A} + \vec{\nabla}\, s)] = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{\nabla}s = \vec{\nabla} \times \vec{A}\,.$
Note: the top and middle of page 196, $\boldsymbol{\nabla} \equiv \vec{\nabla}$, or is it just a typo?
The example presented in Eq. (7) page 196 is used in Exercise 11.3.

Exercise 11.3 The Curl and the Gradient, page 197

Show that the vector potentials in Eqs. (8) page 196, and Eqs. (9) page 197, both give the same uniform magnetic field as presented in Eqs. (7). This means that the two differ by a gradient. Find the scalar whose gradient, when added to Eqs. (8) page 196, gives Eqs. (9) page 197.
Here I present the equations in a table:

Eqs. (7)	Eqs. (8)	Eqs. (9)
$B_x = 0$	$A_x = 0$	$A'_x = -by$
$B_y = 0$	$A_y = bx$	$A'_y = 0$
$B_z = b$	$A_z = 0$	$A'_z = 0$

Now we add the scalar gradient of S to $\vec{A}$:
$(A_x, A_y, A_z) + (\nabla_x S, \nabla_y S, \nabla_z S) = (A'_x, A'_y, A'_z).$
Then,
$A_x + \nabla_x S = A'_x = -by \Longrightarrow \nabla_x S = -by \Longrightarrow S = -byx + C(onstant).$
$A_y + \nabla_y S = 0 \Longrightarrow bx + \nabla_y S = 0 \Longrightarrow \nabla_y S = -bx \Longrightarrow S = -bxy + C(onstant).$
Hence,
$S = -bxy + C \Longrightarrow$ the scalar to be found

11.4 The Force on a Charged Particle, page 198

"*Electrical charged particles are influenced by electric and magnetic fields $\vec{E}$ and $\vec{B}$".*
The force on a charged particle is, the Lorentz force, is presented in Eq. (10) page 199.

11.5 The Lagrangian, page 199

"Question: how to express magnetic forces in the action, or Lagrangian form of mechanics."
On page 200, the possibilities of how to formulate the action in order to give rise to a Lorentz force.
This results into the expression presented at the top of page 201, Eq. (14).

Note:
On page 200 questions arise which notation to be used for the action and the Lagrangian. Why not use $\mathcal{A}$ and $\mathcal{L}$ as before instead of A and L? In the latter case needless confusion is created with the vector potential and momentum. Another issue is, just below the Lagrangian on page 200: *"Here i refers to direction in space, and the summation sign for summing over x, y, z has been left implicit. Get used to it."* On the next pages the summation sign is still there. Why?

On pages 201-203, Eq.(14) page 201, the action is discussed, including the use of a gauge transformation by the gradient of a scalar. This leads to the conclusion this gauge transformation to be invariant: *gauge invariance.*

11.6 Equations of Motion, page 203

The Lagrangian is presented: Eq. (16) page 203, derived from Eq.(14) page 201, the integrand in the action integral:

$\mathcal{L} = \sum_i [\frac{m}{2} (\dot{x}_i)^2 + \frac{e}{c} \dot{x}_i \cdot A_i(x)]$.

The Lagrangian is presented in detail in Eq. (16) page 203.

With Euler-Lagrange,

$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$,

the equation of motion is found.

We have

$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}}$, ,

and

$\dot{p}_x = \frac{\partial \mathcal{L}}{\partial x}$.

With Eq. (16)

$p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} + \frac{e}{c} A_x$, Eq. (18) page 203.

On page 204, the equations of motion are calculated leading to Eq. (20) page 205.

This represents the x-componentof the Newton-Lorentz equation of motion, Eq. (11) page 199.

11.7 The Hamiltonian, page 205

The Hamiltonian

$H = \sum_i p_i \, \dot{q}_i - \mathcal{L}$.

The Hamiltonian is presented in Eq. (22) page 206.

With the Lagrangian

$\mathcal{L} = \sum_i [\frac{m}{2} (\dot{x}_i)^2 + \frac{e}{c} \dot{x}_i \cdot A_i(x)]$,

and

$\dot{x}_i$ presented in Eq. (23) page 207, derived from Eq. (21) page 206,

the Hamiltonian Eq. (24) page 207, is obtained.

Note: Plug x_i

$\dot{x}_i = \frac{1}{m} [p_i - \frac{e}{c} A_i(x)]$, Eq.(23) page 207,

into Eq.(24), page 207,

the familiar Hamiltonian

$H = \frac{1}{2} m v^2$,

has been recovered.

Exercise 11.4: the Hamiltonian and Newton-Lorentz, page 207

Using the Hamiltonian, Eq. (24) page 207, work out Hamilton's equations of motion and show that you just get back the Newton-Lorentz equation of motion.

The Hamiltonian, Eq. (24) page 207,

$H = \sum_i \{\frac{1}{2m}\left[p_i - \frac{e}{c}A_i(x)\right]^2\}$

Hamilton's equations of motion:

$\dot{q}_i = \frac{\partial H}{\partial p_i}$, and $\dot{p}_i = -\frac{\partial H}{\partial q_i}$,

see Eqs. (12) page 155.

Now, I rewrite the Hamiltonian:

$H = \frac{1}{2m}\{\left[p_x - \frac{e}{c}(A_x)\right]^2 + \left[p_y - \frac{e}{c}(A_y)\right]^2 + \left[p_z - \frac{e}{c}(A_z)\right]^2\}.$

$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m}\left[p_x - \frac{e}{c}(A_x)\right].$

$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{m}\left[p_y - \frac{e}{c}(A_y)\right],$

$\dot{z} = \frac{\partial H}{\partial p_z} = \frac{1}{m}\left[p_z - \frac{e}{c}(A_z)\right].$

Then,

$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{e}{mc}\{\left[p_x - \frac{e}{c}(A_x)\right]\frac{\partial A_x}{\partial x} + \left[p_y - \frac{e}{c}(A_y)\right]\frac{\partial A_y}{\partial x} + \left[p_z - \frac{e}{c}(A_z)\right]\frac{\partial A_z}{\partial x},$

$\dot{p}_y = -\frac{\partial H}{\partial y} = \frac{e}{mc}\{\left[p_x - \frac{e}{c}(A_x)\right]\frac{\partial A_x}{\partial y} + \left[p_y - \frac{e}{c}(A_y)\right]\frac{\partial A_y}{\partial y} + \left[p_z - \frac{e}{c}(A_z)\right]\frac{\partial A_z}{\partial y},$

$\dot{p}_z = -\frac{\partial H}{\partial z} = \frac{e}{mc}\{\left[p_x - \frac{e}{c}(A_x)\right]\frac{\partial A_x}{\partial z} + \left[p_y - \frac{e}{c}(A_y)\right]\frac{\partial A_y}{\partial z} + \left[p_z - \frac{e}{c}(A_z)\right]\frac{\partial A_z}{\partial z}.$

Let us look for x-component of the equations of motion.

$\dot{p}_x = -\frac{\partial H}{\partial x} = \frac{e}{mc}\{\left[p_x - \frac{e}{c}(A_x)\right]\frac{\partial A_x}{\partial x} + \left[p_y - \frac{e}{c}(A_y)\right]\frac{\partial A_y}{\partial x} + \left[p_z - \frac{e}{c}(A_z)\right]\frac{\partial A_z}{\partial x}.$

With $\dot{x} = \frac{1}{m}\left[p_x - \frac{e}{c}(A_x)\right], \dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{m}\left[p_y - \frac{e}{c}(A_y)\right]$ and $\dot{z} = \frac{\partial H}{\partial p_z} = \frac{1}{m}\left[p_z - \frac{e}{c}(A_z)\right],$

the latter expression for $\dot{p}_x$ becomes

$\dot{p}_x = \frac{e}{c}(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}).$

Now, plug the preceding expression into the first equation at the top of page 204,

$m\ddot{x} = \dot{p}_x - \frac{e}{c}\left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right),$

resulting into

$m\ddot{x} = \frac{e}{c}\left[\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right) - \left(\frac{\partial A_x}{\partial x}\dot{x} + \frac{\partial A_x}{\partial y}\dot{y} + \frac{\partial A_x}{\partial z}\dot{z}\right)\right] = \frac{e}{c}(\dot{y}\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\dot{y} + \dot{z}\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\dot{z}).$

This expression for $m\ddot{x}$, is equal to Eq. (19) page 204, obtained using the Euler-Lagrange equations.

Hence

$m\ddot{x} = \frac{e}{c}(B_z\dot{y} - B_y\dot{z})$, equal to Eq. (20) page 205.

We just get back the Newton-Lorentz equation of motion.

11.7 Motion in a Uniform Magnetic Field, page 208

The Hamiltonian, Eq. (24) page 207

$H = \frac{1}{2m}\{\left[p_x - \frac{e}{c}(A_x)\right]^2 + \left[p_y - \frac{e}{c}(A_y)\right]^2 + \left[p_z - \frac{e}{c}(A_z)\right]^2\}.$

With Eq. (8),

$A_x = 0, A_y = bx$, and $A_z = 0,$

we obtain for the Hamiltonian

$H = \frac{1}{2m}\{[p_x]^2 + \left[p_y - \frac{e}{c}(bx)\right]^2 + [p_z]^2\}.$

With Eq. (21) page 206,

$p_z = m\dot{y}.$

Conservation of

$p_z : \dot{p}_z = 0 \Longrightarrow \dot{y} = 0.$

With Eq. (21) page 206,
$p_y = m\dot{y} + \frac{e}{c}(bx)$.
Conservation of
$p_y : \dot{p}_y = 0 \Longrightarrow \ddot{y} = -\frac{eb}{mc}\dot{x}$.
Then, Susskind explained p_x not to be conserved due to the dependence of H on x.
The next step is to determine the x-component of acceleration. This is done by using the change in gauge and use Eq. (9)[1] page 197
$A'_x = -by, A'_y = 0$, and $A'_z = 0$.
The Hamiltonian Eq. (24) page 207, with Eq. (9) page 197,
$H = \frac{1}{2m}\{\left[p_x - \frac{e}{c}(A_x)\right]^2 + \left[p_y - \frac{e}{c}(A_y)\right]^2 + \left[p_z - \frac{e}{c}(A_z)\right]^2\}$, becomes
$H = \frac{1}{2m}\{\left[p_x + \frac{e}{c}(by)\right]^2 + \left[p_y\right]^2 + [p_z]^2\}$.
Now, the Hamiltonian depends on y.
The conservation of p_x, with Eq. (21) page 206, gives
$\ddot{x} = \frac{eb}{mc}\dot{y}$, Eq. (26) page 210.

[1] Not Eq. (7) as indicated in the text on page 209. A typo.

Exercise 11.5 Motion in a Uniform Magnetic Field, page 210

Show that in the x, y plane, the solution to Eq. (25) page 209 and Eq. (26) page 210 are a circular orbit with the center of the orbit anywhere on the plane. Find the radius of the orbit in terms of the velocity.

Two equations:

$\ddot{y} = -\frac{eb}{mc}\dot{x}$, Eq. (25),

and

$\ddot{x} = \frac{eb}{mc}\dot{y}$, Eq. (26) .

Then,

$\ddot{x} = \frac{eb}{mc}\ddot{y} = -(\frac{eb}{mc})^2\dot{x} \Longrightarrow \ddot{x} + (\frac{eb}{mc})^2\dot{x} = 0$.

Denote $\frac{eb}{mc} \Longrightarrow \omega$.

The solution for $\dot{x}$, the x-component of the velocity,

$\dot{x} = Ae^{i\omega t}$.

Similarly the y-component of the velocity,

$\dot{y} = Be^{i\omega t}$.

Consequently, integrating both expressions, we obtain the coordinate (x, y) including a unknown coordinate, the two constants of integration. So, the orbit can be anywhere in the plane (x, y).

The radius of the orbit in terms of the velocity:

at $t = 0$,

$A = \dot{x}(0)$, and $B = \dot{y}(0)$.

Furthermore

$\ddot{y} = -\omega\dot{x} = -\omega Ae^{i\omega t}$,

and

$\ddot{y} = i\omega Be^{i\omega t}$.

Hence

$B = iA$.

So,

$\dot{y}(0) = i\dot{x}(0)$.

The, we obtain for the amplitude, the radius of the orbit in terms of the velocity,

$R = \sqrt{|\dot{x}(0)|^2 + |\dot{y}(0)|^2}$,

$R = |\dot{x}(0)|\sqrt{2} = |\dot{y}(0)|\sqrt{2}$

11.8 Gauge Invariance, page 210

A reference is made to quantum mechanics and field theory.

"*Gauge fields and gauge invariance ……….. .They are the central guiding principles that underlie everything, from quantum electrodynamics to general relativity and beyond.*"

Appendix 1, Page 213

A1.1 The Central Force of Gravity, page 213

On page 215, the equation of motion of the earth is presented, Eq. (1).

A1.2 Gravitational Potential Energy, page 215

Knowing the force, given in section on *The Central Force of Gravity,* page 213, and the force is minus the divergence of the potential, then the potential is obtained on page 216;

$V(r) = -\frac{GMm}{r}$,

where M is the mass of the sun and m is the mass of the earth

A1.3 The Earth Moves in a Plane, page 216

The symmetry: rotational symmetry⟹conservation of angular momentum.
The problem to be solved is 2-dimensional.

A1.4 Polar Coordinates, page 217

The coordinates are r, θ.
In Eqs. (2) and (3), page 216, the kinetic and potential energy are presented. See also Lecture 6.

A1.5 Equations of Motion, page 218

The Lagrangian and the Euler-Lagrange equations will do the work.
The Lagrangian in polar coordinates is given in Eq.(4) page 218.
So, the r-coordinate,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\ddot{r},$$

and

$$\frac{\partial \mathcal{L}}{\partial r} = r\dot{\theta}^2 - \frac{GMm}{r^2}\,.$$

Equating these two preceding expressions, the equation of motion Eq. (5) page 218 is found.
For the θ-coordinate

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{d}{dt}mr^2\dot{\theta}\,,$$

and

$$\frac{\partial \mathcal{L}}{\partial \theta} = 0\,.$$

Equating these two preceding equations, the equation of motion Eq. (6) page 219. Is obtained. This equation represents the conservation of momentum.
In the text, this is presented in Eq. (7) page 219.
I would have preferred L_z instead of p_θ.
Momentum $\vec{L} = (0,0,L_z)$.
On the pages 219-221 an effective force, an effective potential and an effective Lagrangian has been derived.
On page 218, the Lagrangian in polar coordinates reads, with Eqs. (2) and (3) page 218,

$$\mathcal{L} = T - V \Rightarrow \mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}.$$

Now, I plug into this expression the result of conservation of momentum, $mr^2\dot{\theta} = L_z$ (a constant),

$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r} = \frac{m}{2}\dot{r}^2 + \frac{L_z^2}{2mr^2} + \frac{GMm}{r}.$$

In Eq. (13) page 221, the effective Lagrangian is presented

$$\mathcal{L}_{effective} = \frac{m}{2}\dot{r}^2 - \frac{L_z^2}{2mr^2} + \frac{GMm}{r}.$$

Well, I consider this a bit confusing:

$$\mathcal{L} - \mathcal{L}_{effective} = \frac{L_z^2}{mr^2},$$

How come?
In the effective Lagrangian the term $\frac{L_z^2}{2mr^2}$, is treated as part of the effective potential. This results in a minus sign. How to keep track of this?
Remark:
The potential is obtained by integrating the force:

$$V = -\int_{-\infty}^{r} F\,dr'\,.$$

A1.6 Effective Potential Energy Diagrams, page 221

Diagrams are instructive out find out about equilibrium points.
This is illustrated in Fig. 4 on page 222. In this Figure the effective potential is plotted.

$V(r) = \frac{L_z^2}{2mr^2} - \frac{GMm}{r}$.

Reminder: the centrifugal force is transferred to the potential energy, Eq. (13) page 221.

A1.7 Kepler's Laws, page 223

The three laws of Kepler are presented on page 223.
The prove the orbit is an ellipse is to be considered a bit difficult and for that reason not proved.
Then, the second law is evaluated, pages 225 (Fig.6) and 226.
The second law: *A line joining a planet and the sun sweeps out equal areas during equal time intervals.* It is about the conservation of momentum in a central force field.
Briefly:

$\vec{L} = \vec{r} \times \vec{p} \Longrightarrow \dot{\vec{L}} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \dot{\vec{p}} = \vec{v} \times \vec{p} + \vec{r} \times \vec{F} = 0 + 0 = 0$. In a central force field $\vec{F}$ is aligned with $\vec{r}$.

The infinitesimal area δA is swept out by an infinitesimal time δt.
The area is:

$\delta A = r\delta\theta \cdot \frac{r}{2} \Longrightarrow \lim_{\delta t \to 0} (\frac{\delta A}{\delta t} = \frac{r^2}{2}\frac{\delta\theta}{\delta t}) = \frac{dA}{dt} = \frac{r^2}{2}\dot{\theta} = \frac{L_z}{2m}$, Eq.(16) page 226

Consequently, $\frac{dA}{dt}$ is a constant.

The third Law: the square of the orbital period of a planet is directly proportional to the cube of the radius of its orbit.
In Lecture 2 the acceleration of a circular orbit is calculated and with Newton's law the third law is proved.

Exercise A1.1 Kepler's third Law, page 227

Show that Eq. (17) is a consequence of Eq. (3) from Lecture 2.
Eq. (17) page 227, the acceleration of an object moving in a circular orbit:

$a = \omega^2 r$.

Lecture 2, page 45 Eq. (3):

$a_x = -r\omega^2 \cos\omega t$,

$a_y = -r\omega^2 \sin\omega t$.

Then,

$a = \sqrt{a_x^2 + a_y^2} = \omega^2 r$.

Equating the gravitational force with mass times this acceleration, we have:

$\omega^2 = \frac{GM}{r^3}$.

Now, ω equals the angular frequency $\Longrightarrow$the period $\tau = \frac{2\pi}{\omega}$.

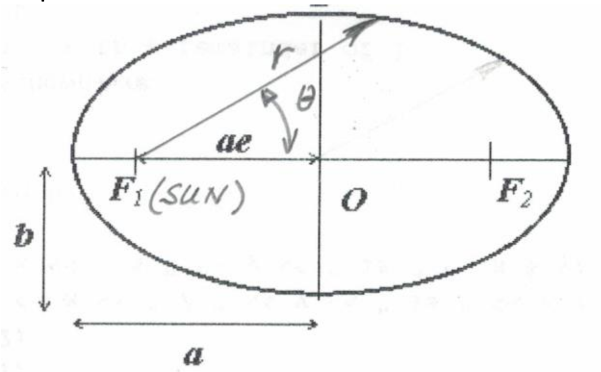
Hence, with $\omega^2 = \frac{GM}{r^3}$, Kepler's law

$\tau^2 = \frac{4\pi^2}{\omega^2} = \frac{4\pi^2}{GM} r^3$.

Appendix 2

A2.1 Proof of Kepler's First Law

See Fowler. The orbit of every planet is an ellipse with the Sun at one of the two foci of the ellipse.



The equation of motion:

$\frac{d^2r}{dt^2} = r\omega^2 - \frac{GM}{r^2}$.

Momentum

$L_z = m\omega r^2 \Longrightarrow \omega = \frac{d\theta}{dt} = \frac{L_z}{mr^2}$.

So,

$\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{L_z^2}{m^2r^3}$.

Change of variable:

$\rho = \frac{1}{r}$.

Another change:

$\frac{d\theta}{dt} = \frac{L_z}{mr^2} \Longrightarrow \frac{d\theta}{dX}\frac{dX}{dt} = \frac{L_z\rho^2}{m} \Longrightarrow \frac{dX}{dt} = \frac{L_z\rho^2}{m}\frac{dX}{d\theta} \Longrightarrow \frac{d}{dt} = \frac{L_z\rho^2}{m}\frac{d}{d\theta}$.

Then,

$\frac{dr}{dt} = \frac{d}{dt}\frac{1}{\rho} = -\frac{L_z}{m}\frac{d}{d\theta}$.

Using the operator $\frac{d}{dt} = \frac{L_z\rho^2}{m}\frac{d}{d\theta}$,

$\frac{d^2r}{dt^2}$ can be written as:

$\frac{d^2r}{dt^2} = -(\frac{L_z}{m})^2\rho^2\frac{d^2\rho}{d\theta^2}$.

For the equation of motion, $\frac{d^2r}{dt^2} = -\frac{GM}{r^2} + \frac{L_z^2}{m^2r^3}$,

we obtain, using the preceding expression,

$\frac{d^2\rho}{d\theta^2} + \rho = GM(\frac{L_z}{m})^2$.

This linear differential equation is a standard nonhomogeneous differential equation. Solution of which can be found in textbooks or the WolframAlpha App.

The solution of the differential equation

$\rho = \frac{1}{r} = GM(\frac{L_z}{m})^2 + A\cos\theta$,

where A is found with the initial condition.

So,

$\frac{1}{r} = GM(\frac{L_z}{m})^2 + A\cos\theta \Longrightarrow \frac{1}{GM\left(\frac{L_z}{m}\right)^2}\frac{1}{r} = 1 + \frac{A}{GM(\frac{L_z}{m})^2}\cos\theta$.

The latter expression resembles the equation for the ellipse in polar coordinates with origin at a focus, the sun:

$\frac{a(1-e^2)}{r} = 1 + e\cos\theta$.

A2.2 Another Proof of Kepler's First Law

https://radio.astro.gla.ac.uk/a1dynamics/ellproof.pdf

The energy equation, in polar coordinates, reads

$$E = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}.$$

The angular momentum of mass m (the earth) is

$$L_z = mr^2\dot{\theta}.$$

Now, we make the same substitution for r as made in section A2.1:

$$\rho = \frac{1}{r}.$$

So,

$$\dot{\theta} = \frac{L_z\rho^2}{m}.$$

The preceding expression results into

$$\theta = \int \frac{L_z\rho^2}{m} dt = \int \frac{L_z\rho^2}{m}\frac{dt}{d\rho} d\rho .$$

With $\dot{r} = -\frac{1}{\rho^2}\frac{d\rho}{dt} \rightarrow \ \rho^2\frac{dt}{d\rho} = -\frac{1}{\dot{r}}$,

then,

$$\theta = \int \frac{L_z\rho^2}{m}\frac{dt}{d\rho} d\rho = -\int \frac{L_z}{m\dot{r}} d\rho.$$

The energy equation can be written as

$$\dot{r}^2 = \frac{2E}{m} + 2GM\rho - (\frac{L_z\rho}{m})^2.$$

Some additional substitution for the constant factors are used

$$r_0 = \frac{L_z^2}{GMm^2},$$

and

$$e^2 = 1 + \frac{2Er_0}{GMm}.$$

Substitute both factors, r_0 and e^2, into the energy equation, $\dot{r}^2 = \frac{2E}{m} + 2GM\rho - \left(\frac{L_z\rho}{m}\right)^2$, resulting after some algebra, into

$$\dot{r} = \frac{L_z}{m}[\left(\frac{e}{r_0}\right)^2 - (\rho - \frac{1}{r_0})^2]^{\frac{1}{2}}.$$

Substitute the preceding expression into

$$\theta = -\int \frac{L_z}{m\dot{r}} d\rho = -\int \frac{1}{[\left(\frac{e}{r_0}\right)^2 - (\rho - \frac{1}{r_0})^2]^{\frac{1}{2}}} d\rho .$$

This is a standard integral. The solution can be obtained from textbooks and/or tables of integrals:

$$\theta = \cos^{-1}(\frac{\rho - \frac{1}{r_0}}{\frac{e}{r_0}}),$$

or

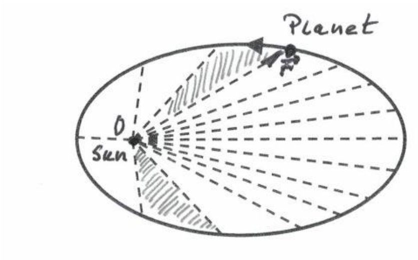
$$r = \frac{r_0}{1 + e\cos\theta},$$

the equation of an ellipse in polar coordinates, with the sun as origin at a focus.

A.2.3 Mathematical Proof of Kepler's Swiped Area Law

- The Swiped Area Law, K2.`

A line joining a planet and the sun sweeps out equal areas during equal time intervals.



It is about a central force field with the sun in the origin of the coordinate system. The vector $\vec{r}$ describes the curve of the planet around the sun.
Some physics: the acceleration $\ddot{\vec{r}}$ is aligned with $\vec{r}$.
Hence
$\vec{r} \times \ddot{\vec{r}} = 0$.
Now, use a curve described by a vector $\vec{n} = \vec{r} \times \dot{\vec{r}}$.
Then,
$\dot{\vec{n}} = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} = 0.$
So,
$\vec{n}$ is perpendicular to the area/plane described by $\vec{r}$ and is a constant vector. The planet moves in a plane through the origin O , Figure above.
Next, we evaluate the small area ΔA swiped by the vector $\vec{r}$ in a timestep Δt. This small area can be approximated by
$\Delta A = \frac{1}{2}|\vec{r} \times [\vec{r}(t+\Delta t) - \vec{r}]|$.
Then the change of the area in the small stime step
$\frac{\Delta A}{\Delta t} = \frac{1}{2}\left|\vec{r} \times \frac{\vec{r}(t+\Delta t)-\vec{r}}{\Delta t}\right| = \frac{1}{2}\left|\vec{r} \times \dot{\vec{r}}\right| = \frac{1}{2} \cdot \vec{n}$, a constant.
The rate of the swept area is constant.
Note, the relation with momentum, $m\dot{\vec{r}}$ (and angular momentum, $m\vec{r} \times \dot{\vec{r}}$) is clearly illustrated, Eq.(16) page 226 Susskind.

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