Quantum Mechanics and the Theoretical Minimum

Updated 2021-08-16 Section 4.8 Ignoring the Phase Factor in Exercise 3.4 and the matrix σ_n some phase factors could not be ignored.

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Remarks, Questions and Exercises.

Based on Quantum Mechanics, The Theoretical Minimum by Susskind, The Feynman Lectures on Physics and The Principles of Quantum Mechanics by Dirac.

Below I adopt the Lecture System of Susskind. The book of Susskind I consider to be a good introduction into the subject matter. Due to the focus on two-state spin particles the book presents a coherent picture.

Contents.

Lecture 1. Systems and Experiment, page 1

1.1 Quantum Mechanics is Different, page 1[1]

Lecture 1 is about Systems and Experiments. In this lecture Susskind explains why quantum mechanics is different. Spins and qubits are introduced illustrated by an experiment which is never gentle. Classical and quantum mechanical propositions are considered. Bras, kets, inner products and orthonormal bases are explained.

1.2 Spins and Qubits, page 3

In the book, spin is paid a lot of attention to.

1.3 An Experiment, page 4

The coin is introduced again. With the coin the idea of state and spin is explained.

1.4 Experiments Are never Gentle, page 12

The basic idea explained is: experiments are always invasive.

1.5 Propositions, page 13

Susskind explained Propositions starting with Boolean logic.

[1] Page numbers of Quantum Mechanics

1.6 Testing Classical Propositions, page 16

A quantum system of a single spin is used for testing proposition.

1.7 Testing Quantum Propositions, page 18

In this section, the logical difference between the classical and quantum concept of the state of a system is discussed.

1.8 Mathematical Interlude: Complex Numbers, page 21

Here Susskind rehearsed the concept of complex numbers.
The vector notation and the Euler notation are given.

1.9 Mathematical Interlude: Vector Spaces, page 24

1.9.1 Axioms, page 24

As explained by Susskind, the space of states in a classical system is a mathematical set. In quantum mechanics the space of states is a vector space.

1.9.2 Functions and Column Vectors, page 27

In this section concrete examples of complex vector spaces are given.

1.9.3 Bras and Kets, page 28

The complex conjugate vector space is introduced.

***Note**: Bras, ⟨|, and kets, |⟩, were invented by Dirac. These represent a remarkably effective and efficient toolkit to deal with the mathematical operations in Quantum Mechanics. Bras mean bra vectors and kets mean ket vectors. Chapter I and II of Dirac give the introduction into ket and bra vectors. To learn more about Dirac I recommend reading of the biography "The Strangest Man, the hidden life of Paul Dirac, quantum genius" by Farmelo. Also, the interview with Dirac in "QED and the men who made it" by Schweber is special.*
For additional reading on the application of bra- and ket vectors see The Feynman Lectures on Physics III. To learn also more about Feynman, I recommend "Genius, The Life and Science of Richard Feynman" by Gleick. In addition, "Surely, You're Joking, Mr Feynman!" edited by Hutchings. And as a follow-up "What Do You Care What Other People Think" by Leighton.

1.9.4 Inner Products, page 30

On top of page 31, Susskind presents:
$\langle B|A\rangle = \langle A|B\rangle^*$.
$\langle B|A\rangle$ is given by row and column vectors and the inner product is calculated:
$\langle B|A\rangle = \sum_i^5 \beta_i^* \alpha_i$, (1.2).
$\langle B|A\rangle^* = \sum_i^5 \beta_i \alpha_i^*$.
With $\langle A|B\rangle^*$
$\langle A|B\rangle = \sum_i^5 \beta_i \alpha_i^*$.
Consequently
$\langle B|A\rangle = \langle A|B\rangle^*$.

Exercise 1.1 About the axioms for inner products, page 31

a) Using the axioms for inner products, prove

$\{\langle A| + \langle B|\}|C\rangle = \langle A|C\rangle + \langle B|C\rangle$. **(L1.1)**

Axioms:

1. They are linear

$\langle C|\{|A\rangle + |B\rangle\} = \langle C|A\rangle + \langle C|B\rangle$

2. Interchanging bras and kets corresponds to complex conjugation:

$\langle B|A\rangle = \langle A|B\rangle^*$

Proof of a):

The complex conjugate of the left-hand side of **(L1.1)**

$$\{\langle A| + \langle B|\}|C\rangle = \langle C|\{|A\rangle + |B\rangle\}^* = \langle C|A\rangle^* + \langle C|B\rangle^* = \langle A|C\rangle + \langle B|C\rangle.$$

b) Prove $\langle A|A\rangle$ is a real number.

Axiom 2 gives $\langle A|A\rangle = \langle A|A\rangle^*$. Suppose $\langle A|A\rangle$ is a complex number. $\langle A|A\rangle^*$ is the complex conjugate. When a complex number equals its complex conjugate, the imaginary part of that complex number must be zero. So $\langle A|A\rangle$ is a real number.

A complex number can be represented by:

$a + ib$,

where $\{a, b \in \mathbb{R}\}$.

With the complex conjugate of a complex number equals the complex number:

$(a + ib)^* = a - ib = a + ib \rightarrow b = 0$.

***Note**: Dirac page 21.*

Exercise 1.2 Application of the axioms for inner products, page 32

Show that the inner product defined by Eq.(1.2)- Eq.(1.2) refers to the book of Susskind- satisfies all the axioms of inner product. The axioms are given in Exercise 1.1.

a). Axiom 1.

$$\gamma_1^*\gamma_2^* \ldots . \gamma_n^* \left\{ \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_n \end{pmatrix} + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_n \end{pmatrix} \right\} = \gamma_1^*\alpha_1 + \gamma_2^*\alpha_2 + \cdots \gamma_n^*\alpha_n + \gamma_1^*\beta_1 + \gamma_2^*\beta_2 + \cdots \gamma_n^*\beta_n$$

and

b). Axiom 2.

$\langle B|A\rangle = \beta_1^*\alpha_1 + \beta_2^*\alpha_2 + \cdots \beta_n^*\alpha_n$,

and

$\langle A|B\rangle^* = \alpha_1^{**}\beta_1^* + \alpha_2^{**}\beta_2^* + \cdots \alpha_n^{**}\beta_n^* = \alpha_1\beta_1^* + \alpha_2\beta_2^* + \cdots \alpha_n\beta_n^*$.

This section is concluded with the definitions of normalized and orthogonal vectors.

1.9.5 Orthonormal Basis, page 32

Remarks:

$|A\rangle = \sum_i \alpha_i |i\rangle$, (Eq. 1.3),

where α_i are complex numbers.

$\langle j|A\rangle = \sum_i \langle j|\alpha_i|i\rangle = \sum_i \alpha_i \langle j|i\rangle$,

where $\langle j|i\rangle = \delta_{ij}$ so $\langle j|A\rangle = \alpha_j$.

Eq. (1.3) can be written in an elegant form. Let us look for this form. One way to obtain this is:

$|A\rangle = \sum_i \alpha_i |i\rangle = \sum_i |i\rangle\alpha_i$ with $\alpha_i = \langle i|A\rangle$ so $|A\rangle = \sum_i |i\rangle\langle i|A\rangle$.

The other way: we assume

$|A\rangle = \sum_i |i\rangle\langle i|A\rangle$ to be correct.

Now $\langle j|A\rangle = \langle j|\sum_i|i\rangle\langle i|A\rangle = \sum_i\langle j|i\rangle\langle i|A\rangle = \langle j|A\rangle$,
since $\langle j|i\rangle = \delta_{ij}$ the Kronecker delta.

Plug Eq. (1.5) into Eq. (1.3):
$|A\rangle = \sum_i\langle i|A\rangle\,|i\rangle$.
This expression rewrites Susskind into
$|A\rangle = \sum_i|i\rangle\langle i|A\rangle$.
Elegant indeed, why? Well, it is basically about the outer product. We will learn about the outer product in section 7.2.

Lecture 2. Quantum States.

States and vectors are discussed. Spin states along the x-axis, the y-axis and the z-axis are given attention. The number of independent parameters and the representation of spin states as column vectors are introduced.

2.1 States and Vectors

***Note**: This lecture starts with a reflection on the unpredictability and the completeness of Quantum Mechanics. Susskind adopts the unpredictability of QM. Of course, there is no ultimate answer. An interesting view on the subject matter is given by Stewart: "Does God Play Dice?". It is all about hidden variables. Also, the book of Smolin is instructive reading in this respect: when you need decades of constants to adjust theory to explain experiment you could wonder about the need of hidden variables.*

2.2 Representing Spin States

As mentioned by Susskind the space of states for a single spin has only two dimensions.
$|u\rangle$ and $|d\rangle$ are chosen as the two orthonormal basic vectors.
All possible spin states can be represented in a two-dimensional vector space. Top of page 38.
With
$A = \alpha_u|u\rangle + \alpha_d|d\rangle \rightarrow$
$\rightarrow \langle u|A\rangle = \langle u|\alpha_u|u\rangle + \langle u|\alpha_d|d\rangle = \alpha_u\langle u|u\rangle + \alpha_d\langle u|d\rangle$.
With $\langle u|d\rangle = 0$, Eq. (2.3),
$\alpha_u = \langle u|A\rangle$.
The state vector is normalized: $\langle A|A\rangle = 1$, consequently
$(\langle u|\alpha_u^* + \langle d|\alpha_d^*)(\alpha_u|u\rangle + \alpha_d|d\rangle) = \alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$.
$\alpha_u^*\alpha_u$ and $\alpha_d^*\alpha_d$ represent probabilities. Consequently, in this two base vector system, these probabilities add up to 1: (2.2).

2.3 Along the x Axis

Then, Susskind derived the vectors $|r\rangle$ and $|l\rangle$ along the x – axis presented by Eq. 2.5 and Eq. 2.6. After deriving vector $|r\rangle$, one must remember that $\langle l|r\rangle = 0$ (the inner product) and
$\alpha_u^*\alpha_u = \frac{1}{2}$ and $\alpha_d^*\alpha_d = \frac{1}{2}$. **(L2.1)**
These equal probabilities are explained by Susskind just above Eq. 2.5.
The α's are complex numbers, and we can illustrate phase ambiguity by representing the

complex numbers in polar coordinates(q, θ):
$\alpha_u = q_u e^{i\theta_u}$ and $\alpha_d = q e^{i\theta_d}$,
With **(L2.1)** we find $q_u = \pm\frac{1}{\sqrt{2}}$ and $q_d = \pm\frac{1}{\sqrt{2}}$, where the minus sign represents a phase shift of π. On page 42 Susskind explains why you can neglect the phase factor $e^{i\theta}$.
Now $|l\rangle$: with the same procedure $|l\rangle = \beta_u|u\rangle + \beta_d|d\rangle$ and polar coordinates we have:
$\beta_u = \frac{1}{\sqrt{2}}$ and $\beta_d = -\frac{1}{\sqrt{2}}$.
where use has been made of $\langle l|r\rangle = 0$.
There is no other way to find out about the coefficients used for the vectors $|r\rangle$ and $|l\rangle$.
We have 3 equations and 4 unknowns:
$|r\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle \Longrightarrow \alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1,$
$|l\rangle = \beta_u|u\rangle + \beta_d|d\rangle \Longrightarrow \beta_u^*\beta_u + \beta_d^*\beta_d = 1,$
$\langle l|r\rangle = 0 \Longrightarrow \beta_u^*\alpha_u + \beta_d^*\alpha_d = 0.$

Exercise 2.1 About orthogonality
Prove that the vector $|r\rangle$ in Eq. 2.5 is orthogonal to vector $|l\rangle$ in Eq. 2.6:
$\langle l|r\rangle = (\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}|u\rangle - \frac{1}{\sqrt{2}}|d\rangle = \frac{1}{2}\langle u|u\rangle - \frac{1}{2}\langle u|d\rangle + \frac{1}{2}\langle d|u\rangle - \frac{1}{2}\langle d|d\rangle = 0.$

2.4 Along the y Axis

About vectors representing spins oriented along the y axis.
Remark:
Susskind used $|i\rangle$ as a vector for the y-axis. Do not confuse this with $i = \sqrt{-1}$.
Susskind presents the conditions for the vectors representing the spins along the y-axis. From the statistical results of the experiments, Eqs.2.8 and 2.9 are derived. The conditions presented in these equations are sufficient to derive Eq.2.10.

Exercise 2.2 About Probabilities
Prove that $|i\rangle$ and $|o\rangle$ satisfy all the conditions in Eqs. 2.7, 2.8, and 2.9. Are they unique in that respect?
Eq. 2.7 $\langle i|o\rangle = 0$. $\langle i|o\rangle = (\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{-i}{\sqrt{2}})(\frac{1}{\sqrt{2}}|u\rangle - \frac{i}{\sqrt{2}}|d\rangle) = \frac{1}{2} - \frac{1}{2} = 0.$
Eq.2.8 As an example we take $P_u =< i|u >< u|i >= \frac{1}{2}$.
So $\left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}\right)|u\rangle\langle u|\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle\right) = \frac{1}{2}.$
Eq. 2.9 As an example $\langle i|r\rangle\langle r|i\rangle = \frac{1}{2}$. Here we have $|r\rangle$ instead of $|u\rangle$. A bit more complicated but straight forward.
$\left(\langle u|\frac{1}{\sqrt{2}} - \langle d|\frac{i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{1}{\sqrt{2}}|d\rangle\right)\left(\langle u|\frac{1}{\sqrt{2}} + \langle d|\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}|u\rangle + \frac{i}{\sqrt{2}}|d\rangle\right) = \frac{1}{2}.$

Are the vectors $|i\rangle$ and $|o\rangle$ unique? No, they are not. There is phase ambiguity and the α's are complex.
Note: Dirac page 22.
By setting the phase factor equal to 0 and equal to $\frac{\pi}{2}$, the α's are real or pure imaginary, respectively. I will show this in the next exercise.

Exercise 2.3 More about the components of |i> and |o>

For the moment, forget that Eqs. 2.10 give us working definitions for $|i\rangle$ and $|o\rangle$ in terms of $|u\rangle$ and $|d\rangle$, and assume the components α, β, γ and δ are unknown:

$|i\rangle = \alpha|u\rangle + \beta|d\rangle,$ **(L2.2)**

$|o\rangle = \gamma|u\rangle + \delta|d\rangle.$ **(L2.3)**

a) Use Eqs. 2.8 to show that $\alpha^*\alpha = \beta^*\beta = \gamma^*\gamma = \delta^*\delta = \frac{1}{2}$.

With **(L2.2)** and **(L2.3)**:

$\langle o|u\rangle = (\langle u|\gamma^* + \langle d|\delta^*)|u\rangle = \gamma^*$ and $\langle u|o\rangle = \langle u|(\gamma|u\rangle + \delta|d\rangle) = \gamma,$

where use has been made of: $\langle u|\gamma^*|u\rangle$ and $\langle d|\delta^*|u\rangle = 0$.

Furthermore, Eqs. (2.8), $\langle o|u\rangle\langle u|o\rangle = \frac{1}{2}$, and consequently $\gamma^*\gamma = \frac{1}{2}$.

Analogously we obtain $\alpha^*\alpha = \beta^*\beta = \delta^*\delta = \frac{1}{2}$.

b) With the result of **a)** and Eqs. 2.9 show that $\alpha^*\beta + \alpha\beta^* = \gamma^*\delta + \gamma\delta^* = 0$.

We know, Eqs. (2.9), that $\langle i|r\rangle\langle r|i\rangle = \frac{1}{2}$.

Then, with $|r\rangle$ given in Eq. (2.5) and $|i\rangle$ given in **(L2.2),** we have

$\langle i|r\rangle\langle r|i\rangle = \frac{1}{2}(\alpha^* + \beta^*)(\alpha + \beta) = \frac{1}{2}.$

Consequently, $(\alpha^* + \beta^*)(\alpha + \beta) = 1.$

From the preceding expression we learn the real part $\alpha^*\alpha + \beta^*\beta = 1$, and the complex part $\alpha^*\beta + \alpha\beta^* = 0$.

We could have used the results under **a),** $\langle i|u\rangle\langle u|i\rangle = \alpha^*\alpha = \frac{1}{2}$ and $\langle i|d\rangle\langle d|i\rangle = \beta^*\beta = \frac{1}{2}$.

Then, the result of $\langle i|r\rangle\langle r|i\rangle$ gives again: $\alpha^*\beta + \alpha\beta^* = 0.$

In the same way we obtain, with $\langle o|r\rangle\langle r|o\rangle = \frac{1}{2}$:

$\gamma^*\delta + \gamma\delta^* = 0.$

c) Show that $\alpha^*\beta$ and $\gamma^*\delta$ must each be pure imaginary.

We have shown in **a)** that $\alpha^*\alpha = \beta^*\beta = \frac{1}{2}$. Hence $\alpha^*\beta\alpha\beta^* = \frac{1}{4}$.

Note: set $\alpha = a + ib$ and $\beta = c + id$, where $\{a, b, c, d \in \mathbb{R}\}$.

Then $\alpha^*\alpha\beta^*\beta = \alpha^*\beta\alpha\beta^*$.

Furthermore with **b)** :

$\alpha\beta^* = -\alpha^*\beta$ we find $(\alpha^*\beta)^2 = -\frac{1}{4}$.

Hence, $\alpha^*\beta$ is pure imaginary. Analogously $\gamma^*\delta$ is pure imaginary. This leads to the conclusion, in general, α, β, γ and δ to be complex numbers. Now we have phase ambiguity, and we can choose α, γ to be real and β, δ to be imaginary.

This can be illustrated a little more elegantly with polar coordinates.

For α we write: $\alpha = r_u e^{i\theta_u}$ and $\beta = r_d e^{i\theta_d}$.

With **a)** $\alpha^*\alpha = \frac{1}{2}$: $r_u = \frac{1}{\sqrt{2}}$ and similarly $r_d = \frac{1}{\sqrt{2}}$.

Then with $(\alpha^*\beta)^2 = -\frac{1}{4}$ we have $e^{2i(\theta_d - \theta_u)} = -1 = e^{i\pi}$. So $\theta_d - \theta_u = \frac{\pi}{2}$, a sort of relative phase factor.

Now we choose arbitrarily $\theta_u = 0$ consequently $\theta_d = \frac{\pi}{2}$. Hence α is real and β is imaginary. The same reasoning applies to γ and δ.

2.5 Counting Parameters.

What are the number of physically distinct states for a spin?

Susskind explained the need of just two parameters to specify the spin.

2.6 Representing Spin States as Column Vectors.

In this section Susskind presented an example of Ockham's razor: "…. *and we'll try to choose the simplest and most convenient ones we can find*." With "*ones*" Susskind meant the column vector representation of the bra's and kets.

2.7 Putting It All Together

The title of this section reflects the content of this section: a summary of Chapter 2.
This includes a remark on phase indifference: "*The physics of the state-vector is unaffected by its overall phase factor*". Exercise 2.3 shows that the relative phase factor cannot be ignored.

Dirac: "Any state of our dynamical system at a particular time can be specified by the direction of a bra as well as by the direction of a ket vector. The whole theory will be symmetrical in its essentials between bras and kets.

Lecture 3. Principles of Quantum Mechanics.

Lecture 3 is about the Principles of Quantum Mechanics, (Hermitian)operators, eigenvalues, and eigenvectors. Constructing spin operators is the core of this lecture. The lecture concludes with the Spin-Polarization Principle.

3.1 Mathematical Interlude: Linear Operators

3.1.1 Machines and Matrices

In this section linear operators are introduced: Physical observables are described by linear operators. Throughout the Lectures, Susskind denote a general operator with $\boldsymbol{M}$, for obvious reasons. Properties of $\boldsymbol{M}$ are presented.
The operator $\boldsymbol{M}$ as a matrix is introduced, Eq.(3.2), and $m_{kj} = \langle k|\boldsymbol{M}|j\rangle$ are the matrix elements.
The way the operator $\boldsymbol{M}$ operates on kets and bra's is explained in some detail.
On page 54, this is shown with
$\boldsymbol{M}|A\rangle = |B\rangle$,
this expression is written in component form.
$|A\rangle = \sum_j \alpha_j |j\rangle$.
$|B\rangle$ is expanded in the same basis:
$|B\rangle = \sum_j \beta_j |j\rangle$.
So,
$\boldsymbol{M}|A\rangle = |B\rangle \rightarrow \sum_j \boldsymbol{M}\alpha_j |j\rangle = \sum_j \boldsymbol{M}|j\rangle \alpha_j = \sum_j \beta_j |j\rangle$.
Take the inner product of both sides with the bra of one of the basis vectors $|k\rangle$:
$\sum_j \langle k|\boldsymbol{M}|j\rangle \alpha_j = \sum_j \beta_j \langle k|j\rangle$, Eq.(3.1),
where $\langle k|j\rangle = \delta_{kj}$.
For $k \neq j \rightarrow \delta_{kj} = 0$, and for $k = j \rightarrow \delta_{kj} = 1$; the Kronecker delta.
Then, with Eq.(3.1) and $k = j$
$\langle k|\boldsymbol{M}|k\rangle \alpha_k = \beta_k$.
For $k \neq j$, with Eq.(3.1)
$\sum_j \langle k|\boldsymbol{M}|j\rangle \alpha_j = 0$.
Now, Eq.(3.3), page 55,
$\sum_j m_{kj}\alpha_j = \beta_k$.
This result differs from $\langle k|\boldsymbol{M}|k\rangle \alpha_k = \beta_k$, unless matrix $\boldsymbol{M}$ is a matrix with only diagonal entries:

$\langle k|\boldsymbol{M}|j\rangle\alpha_j = 0$, for $k \neq j$.
So, the analysis on pages 54 and 55 remains a bit unclear to me.

3.1.2 Eigenvalues and Eigenvectors

The definition of the eigenvector and eigenvalues of a linear operator $\boldsymbol{M}$ are presented, Eq.(3.5).
An example:
$\boldsymbol{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
Then,
$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ i \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ i \end{pmatrix} \Rightarrow \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} \lambda \\ i\lambda \end{pmatrix} \Rightarrow \lambda = -i \Rightarrow i\lambda = 1.$
Hence, the eigenvalue $\lambda = -i$.

3.1.3 Hermitian Conjugation

In section 3.1.2 the way $\boldsymbol{M}$ operates on bra's and kets was explained in some detail. There is more. To operate $\boldsymbol{M}$ on bra's, complex conjugation is needed.
Then, Susskind explained the concept of complex conjugation and the definition of Hermitian conjugate is introduced.
On page 61, Susskind presents the consequences of the Hermitian operator $\boldsymbol{M}$:
$\boldsymbol{M}|A\rangle = |B\rangle$,
then
$\langle A|\boldsymbol{M}^\dagger = \langle B|$.
Then, multiply $\boldsymbol{M}|A\rangle = |B\rangle$ to the left with the bra $\langle B|$, and multiply $\langle A|\boldsymbol{M}^\dagger = \langle B|$ to the right with the ket $|B\rangle$,
$\langle B|\boldsymbol{M}|A\rangle = \langle B|B\rangle = \langle A|\boldsymbol{M}^\dagger|B\rangle.$ (L.3.1.3.1)
I recall the expression on top of page 31:
$\langle B|A\rangle = \langle A|B\rangle^*.$ (L.3.1.3.2)
Plug $\langle A|\boldsymbol{M}^\dagger = \langle B|$, and $\boldsymbol{M}|A\rangle = |B\rangle$, into L(3.1.3.2)
$\langle A|\boldsymbol{M}^\dagger|A\rangle = \langle A|\boldsymbol{M}|A\rangle^*.$
Assume the kets and bras to be normalized: $\langle A|A\rangle = 1$ and $\langle B|B\rangle = 1$.
Plug into (L.3.1.3.1), in $\langle B|\boldsymbol{M}|A\rangle$, $\langle A|\boldsymbol{M}^\dagger = \langle B|$, then
$\langle A|\boldsymbol{M}^\dagger\boldsymbol{M}|A\rangle = \langle B|B\rangle = 1$. (L.3.1.3.3)
In addition:
$\langle A|A\rangle = 1.$
What kind of operator does $\boldsymbol{M}^\dagger\boldsymbol{M}$ represent?
Plug into (L.3.1.3.2), $|B\rangle = \boldsymbol{M}|A\rangle$ and $\langle A|\boldsymbol{M}^\dagger = \langle B|$.
The result is,
$\langle A|\boldsymbol{M}^\dagger|A\rangle = \langle A|\boldsymbol{M}|A\rangle^*.$ (L.3.1.3.4)
What kind of operator is $\boldsymbol{M}$? A general operator? It should be. (L.3.1.3.4) most probably represents an identity.

3.1.4 Hermitian Operators

In this Lecture Susskind writes: "*Real numbers play a special role in physics. The results of any measurement are real numbers. Sometimes, we measure two quantities, put them together with an i (forming a complex number) and call this number the result of a measurement*".

Dirac page 35: " One might think one could measure a complex dynamical variable by measuring separately its real and pure imaginary part. But this would involve two measurements or two observations, which would be all right in classical mechanics, but would not do in quantum mechanics, where two observations in general interfere with one another- it is not in general permissible to consider that two observations can be made exactly simultaneously, and if they are made in quick succession the first will usually disturb the state of the system and introduce an indeterminacy that will affect the second".

Question: Will a dialogue Dirac-Susskind help, a dialogue like the one between Democritus and Lederman(Lederman)?

Hermitian operators and the related observables are introduced.
Susskind proved the eigenvalues of Hermitian operators to be real.

3.1.5 Hermitian Operators and Orthonormal Bases

Hermitian operators and orthonormal bases are discussed.
On page 66 Susskind mentioned the possibility of constructing an orthonormal basis. He illustrated this with a two-dimensional case. Then he writes: *"It should be clear that any linear combination of the two eigenvectors is also an eigenvector with the same eigenvalue. With this much freedom, it is always possible to find two orthonormal linear combinations."*
Well, I think referring to this two-dimensional case one linear combination is sufficient. This linear combination is constructed with help of a combination of the vectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$. Or is this what Susskind meant by two linear combinations?
Or another possibility could be:

$$|A\rangle = \alpha|\lambda_1\rangle + \beta|\lambda_2\rangle, \qquad \text{(L.3.1.5.1)}$$

$$|B\rangle = \gamma|\lambda_1\rangle + \delta|\lambda_2\rangle. \qquad \text{(L.3.1.5.2)}$$

Now,
$\langle A|B\rangle$ can be 0? Well, α, β, γ and δ are complex numbers. Analyse the result of the inner product $\langle A|B\rangle$ by representing the complex numbers by a real and an imaginary part.
See Exercise 3.1 below.
First, let's analyse
$\langle A|A\rangle \rightarrow |\alpha|^2 + |\beta|^2 = 1.$
$\langle A|\lambda_1\rangle\langle\lambda_1|A\rangle = |\alpha|^2.$
$\langle A|\lambda_2\rangle\langle\lambda_2|A\rangle = |\beta|^2.$
Then

$$\langle A|A\rangle = \langle A|\lambda_1\rangle\langle\lambda_1|A\rangle + \langle A|\lambda_2\rangle\langle\lambda_2|A\rangle. \qquad \text{(L.3.1.5.3)}$$

Hence,
$\langle A|A\rangle = \langle A(|\lambda_1\rangle\langle\lambda_1| + |\lambda_2\rangle\langle\lambda_2|)A\rangle.$
Consequently

$$|\lambda_1\rangle\langle\lambda_1| + |\lambda_2\rangle\langle\lambda_2| = 1. \qquad \text{(L.3.1.5.4)}$$

What kind of machine is represented in (L.3.1.5.4)?

Well, in general (L.3.1.5.4) represents

$\sum_i |\lambda_i\rangle\langle\lambda_i| = 1,$ (L.3.1.5.5)

or

$\sum_i |i\rangle\langle i| = 1,$

see pages 33 and 34.

$|A\rangle = \sum_i |i\rangle\langle i|A\rangle,$

where "$|i\rangle$ is the orthonormal basis of ket vectors labelled $|i\rangle$."

$\langle A|A\rangle = 1 = \langle A|\sum_i |i\rangle\langle i|A\rangle \rightarrow \sum_i |i\rangle\langle i| = 1.$ (L.3.1.5.6)

Next: the Exercise 3.1.

Exercise 3.1: About an orthonormal base

Prove the following: If a vector space is N-dimensional, an orthonormal basis of N vectors can be constructed from the eigenvectors (plural, > 1) of a Hermitian operator.

If a space is N-dimensional there will be N orthonormal vectors.

I suppose The Fundamental Theorem on page 64 does apply here.

These orthonormal vectors can be constructed of the eigenvectors of a Hermitian operator. See page 66.

With two orthonormal eigenvectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$ of a Hermitian operator, I choose two linear combinations

$|A\rangle = \alpha|\lambda_1\rangle + \beta|\lambda_2\rangle,$ (L3.1.5.1),

and

$|B\rangle = \gamma|\lambda_1\rangle + \delta|\lambda_2\rangle,$ (L3.1.5.2).

Now,

$\langle A|B\rangle = 0.$

Then,

$\alpha^*\gamma + \beta^*\delta = 0\ .$ (L.3.1.5.7)

$\langle A|A\rangle \rightarrow |\alpha|^2 + |\beta|^2 = 1,$ (L.3.1.5.8)

and

$\langle B|B\rangle \rightarrow |\gamma|^2 + |\delta|^2 = 1.$ (L.3.1.5.9)

(L.3.1.5.7)- (L.3.1.5.9): four equations and four unknowns → $|A\rangle$ and $|B\rangle$ can be constructed. Four equations, since (L.3.1.5.7) produces two equations.

3.1.6 The Gram-Schmidt Procedure

In this section Susskind explains the Gram-Schmidt Procedure. What seems to me a bit confusing is an inner product dealt with as a vector. See Figure 3.1, Legenda.

So, in the legenda, the expression for $\vec{V}_{2\perp}$ should read

$\vec{V}_{2\perp} = \vec{V}_2 - \langle\vec{V}_2|\vec{V}_2\rangle\hat{v}_1\ .$

I think this to be a bit more elegant or correct.

Remark: In Lecture 7.2 another approach for the Gram-Schmidt procedure is presented.

3.2 The Principles

Susskind states The Principles of Quantum Mechanics(By the way the title of Dirac's book) .

"*An important consequence of the principles is as follows: The operators that represent observables are Hermitian".* With help of the proof by contradiction and section 3.1.5 this can be shown.

Let us proof it anyway. With the eigenvalues of the operator $\boldsymbol{L}$ and the eigenvectors, with different eigenvalues, orthogonal, $\boldsymbol{L}$ must be Hermitian.

Proof

We presume $\boldsymbol{L} \neq \boldsymbol{L}^{\dagger}$ and follow the notation of page 65 we write:

$\boldsymbol{L}|\lambda_1\rangle = \lambda_1|\lambda_1\rangle$, and $\boldsymbol{L}|\lambda_2\rangle = \lambda_2|\lambda_2\rangle$.

Now, create the complex conjugate of $\boldsymbol{L}|\lambda_1\rangle = \lambda_1|\lambda_1\rangle$:

$\langle\lambda_1|\boldsymbol{L}^{\dagger} = \lambda_1\langle\lambda_1|$, (L.3.2.1)

keep in mind: use has been made of results of an experiment to be real numbers. Consequently, eigenvalue of the operator $\boldsymbol{L}$ must be a real number(Susskind page 74).

Now, $\boldsymbol{L}|\lambda_2\rangle = \lambda_2|\lambda_2\rangle$.

Construct the inner product of $\langle\lambda_1|\boldsymbol{L}^{\dagger} = \lambda_1\langle\lambda_1|$, (L.3.2.1), and $|\lambda_2\rangle$:

$\langle\lambda_1|\boldsymbol{L}^{\dagger}|\lambda_2\rangle = \lambda_1\langle\lambda_1|\lambda_2\rangle$. (L.3.2.2)

Then, $\boldsymbol{L}|\lambda_2\rangle = \lambda_2|\lambda_2\rangle$, and form its inner product with $\langle\lambda_1|$:

$\langle\lambda_1|\boldsymbol{L}|\lambda_2\rangle = \lambda_2\langle\lambda_1|\lambda_2\rangle$. (L.3.2.3)

Subtract (L.3.2.2) and (L.3.2.3):

$\langle\lambda_1|\boldsymbol{L}^{\dagger}|\lambda_2\rangle - \langle\lambda_1|\boldsymbol{L}|\lambda_2\rangle = \lambda_1\langle\lambda_1|\lambda_2\rangle - \lambda_2\langle\lambda_1|\lambda_2\rangle$. (L.3.2.4)

Since the eigenvectors are orthogonal, the right-hand sight of (L.3.2.4) is zero.

Then, $\langle\lambda_1|\boldsymbol{L}^{\dagger} - \boldsymbol{L}|\lambda_2\rangle = 0$. Consequently $\boldsymbol{L} = \boldsymbol{L}^{\dagger} \rightarrow \boldsymbol{L}$ is Hermitian.

End of Proof.

Remark:

This is just a part of the proof? Do we have to proof that the set of eigenvectors is complete? Well, I consider this not to be that easy, though Susskind states otherwise on page 67: "*The final part of the theorem states that the eigenvectors are complete. In other words, if the space is N-dimensional, there will be N orthonormal eigenvectors. The proof is easy*".

May the statement be not quite clear with respect to which proof is easy: the completeness or the N orthonormal eigen vectors or both? Above I focused on the proof of the operator to be Hermitian. Having done that, let us try to find an orthonormal base for a NxN Hermitian operator with for example two eigenvectors with equal eigenvalues. Consequently, we have $N-1$ different eigenvalues with eigenvectors orthonormal. We assume two eigenvectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$ to have the same eigenvalue λ. Can we find a new eigenvector normal to $|\lambda_2\rangle$?

We follow Susskind. Consider an arbitrary linear combination of $|\lambda_1\rangle$ and $|\lambda_2\rangle$:

$|A\rangle = \alpha|\lambda_1\rangle + \beta|\lambda_2\rangle$, α and $\beta \neq 0$. There are two conditions: $\langle\lambda_2|A\rangle = 0$ and $\langle A|A\rangle = 1$. The first condition gives us:

$\alpha\langle\lambda_2|\lambda_1\rangle + \beta\langle\lambda_2|\lambda_2\rangle = \alpha\langle\lambda_2|\lambda_1\rangle + \beta = 0.$

So,

$\langle\lambda_2|\lambda_1\rangle = \frac{-\beta}{\alpha} = \frac{-\beta\alpha^*}{\alpha\alpha^*}$, then $\langle\lambda_1|\lambda_2\rangle = \frac{-\alpha\beta^*}{\alpha\alpha^*}$. Reminder: $\alpha\alpha^* = |\alpha|^2$.

The second condition gives us:

$(\langle\lambda_1|\alpha^* + \langle\lambda_2|\beta^*)(\alpha|\lambda_1\rangle + \beta|\lambda_2\rangle) = \alpha^*\alpha + \beta^*\beta + \alpha^*\beta\langle\lambda_1|\lambda_2\rangle + \beta^*\alpha\langle\lambda_2|\lambda_1\rangle = 1.$

With the expressions for the inner products $\langle\lambda_2|\lambda_1\rangle$ and $\langle\lambda_1|\lambda_2\rangle$, derived above, this expression becomes:

$|\alpha|^2 - |\beta|^2 = 1$. We find for α: $|\alpha| = \sqrt{1 + |\beta|^2}$. Is this to expected? Yes, it is.

I start with $|\lambda_1\rangle$ and $|\lambda_2\rangle$ to be orthonormal and construct a vector

$|A\rangle = \alpha|\lambda_1\rangle + \beta|\lambda_2\rangle.$

For this $|A\rangle$ to be normalized we have:

$|\alpha|^2 + |\beta|^2 = 1,$

and

$|\alpha|$ and $|\beta|$ both smaller than 1.

To visualise this, take two orthonormal basic vectors. Construct the diagonal and normalize

the diagonal vector. To this end, a multiplication factor of $\frac{1}{2}\sqrt{2}(<1)$ is needed.
In the case of degeneracy, I need to construct two mutually orthogonal eigenstates $|A\rangle$ and $|\lambda_2\rangle$. Furthermore, $\langle A|A\rangle = 1$, $|\alpha|$ and $|\beta|$ both must be larger than 1. Hence, $|\alpha| = \sqrt{1+|\beta|^2}$ is correct.
To visualise this case, take two normalized base vectors having an angle of $\frac{\pi}{4}$, i.e. $\langle\lambda_2|\lambda_1\rangle = \frac{1}{2}\sqrt{2}$. To construct a normalized vector $|A\rangle$, a multiplication factor $\sqrt{2}(>1)$ is needed.
Well, let's investigate the subject matter in a different way.
I found above:
$\langle\lambda_2|\lambda_1\rangle = \frac{-\beta}{\alpha}$.
Then
$|A\rangle = \alpha|\lambda_1\rangle + \beta|\lambda_2\rangle = \alpha(|\lambda_1\rangle - \langle\lambda_2|\lambda_1\rangle|\lambda_2\rangle).$ **(L.3.2.5)**
Hence,
$\langle A|A\rangle = |\alpha|^2(1 - |\langle\lambda_2|\lambda_1\rangle|^2)$, **(L.3.2.6)**
where use has been made of $\langle\lambda_1|\lambda_1\rangle = 1$ and $\langle\lambda_2|\lambda_2\rangle = 1$.
Consequently:
$|\alpha| = \frac{1}{\sqrt{1-|\langle\lambda_2|\lambda_1\rangle|^2}}$.

Remark:$|\alpha| > 1$. As mentioned above, for $\langle\lambda_2|\lambda_1\rangle = \frac{1}{2}\sqrt{2}$, $|\alpha| = \sqrt{2}$.
Plug $|\alpha|$ into **(L.3.2.5)**
$|A\rangle = \frac{|\lambda_1\rangle - \langle\lambda_2|\lambda_1\rangle|\lambda_2\rangle}{\sqrt{1-|\langle\lambda_2|\lambda_1\rangle|^2}}$, **(L.3.2.7)**
where, without loss of generality, an arbitrary phase angle is neglected in α .
Multiply the bra $\langle\lambda_2|$ into **(L.3.2.7)** and the orthogonality of $|A\rangle$ and $|\lambda_2\rangle$ is found. $|A\rangle$ has been elegantly expressed in the basic vectors $|\lambda_1\rangle$ and $|\lambda_2\rangle$.
With the Gram-Schmidt Procedure the same result is obtained. Then,
$|A\rangle = |\lambda_1\rangle - \langle\lambda_2|\lambda_1\rangle|\lambda_2\rangle.$ **(L.3.2.8)**
To find $\langle A|A\rangle = 1$, I must divide **(L.3.2.8)** by $\sqrt{1-|\langle\lambda_2|\lambda_1\rangle|^2}$.

We can expand into $k(\geq 3)$ eigenvectors with the same eigenvalue and consequently $N-k$ eigenvectors with different eigenvalues. In words: For a vector space of dimension N we have a NxN Hermitian operator. Suppose we have $N-k$ orthonormal eigenvectors with different eigenvalues and k eigenvectors with one and the same eigenvalue. Then we can find k orthonormal eigenvectors by writing these vectors as linear combinations of the set of k eigenvectors which span the subspace of eigenvectors of the Hermitian operator with the same eigenvalue.
Question: An important question, at least to me, is: To what purpose do we need to create a complete set of orthonormal eigenvectors? After completion of the set of eigenvectors the eigenvalues are still the same. Well, to represent a general ket vector you need the complete set. Think about three dimensional ket space. Using two basic vector means a two-dimensional ket instead of the general three-dimensional general ket.

Eigenvalues are important according to Principle 2(Susskind page 70): *"The possible results of a measurement are the eigenvalues of the operator that represents the observable."* On page 74 Susskind writes:" ……*Secondly, the eigenvectors that represent unambiguously distinguishable results must have different eigenvalues and must be orthogonal.*" So, the above question will not go away.
Dirac describes observables in paragraph 10. There he mentioned you need a complete set of orthonormal eigenstates(vectors). That does not answer my question. Having completed the orthonormal set, no new eigenvalue is found.

***Note**: Dirac, on page 32, proofs the theorem "Two eigenvectors of a real dynamical variable belonging to different eigenvalues are orthogonal". On page 30 one can read the following statement: "The theory of eigenvalues and eigenvectors of a linear operator α which is not real is not of much use for quantum mechanics". On page 32 Dirac presents the proof that an arbitrary ket can be expressed as the sum of eigen kets of a real linear operator $\boldsymbol{L}$ (notation Susskind) which satisfies an algebraic equation: $\phi(\boldsymbol{L}) \equiv \sum_{k=0}^{\boldsymbol{n}} a_{\boldsymbol{k}} \boldsymbol{L}^{n-k} = 0$.*

Principle 2: The possible results of a measurement are the eigenvalues of the operator that represents the observable, page 70. Susskind writes that he will flesh these Principle out, since this statement is hardly self-explanatory. I do not know what fleshing out really means.
On page 73, Principle 4 is explored. There $\langle A|\lambda_i\rangle = \langle\lambda_i|A\rangle^*$ has been used.
$P(\lambda_i)$ is the probability of measuring λ_i. Any measurement creates a λ_i.
Hence,
$P = \sum_i P(\lambda_i) = 1.$
Back to page 34:
$|A\rangle = \sum_{\lambda_i} |\lambda_i\rangle\langle\lambda_i|A\rangle.$
Then
$\langle A| = \sum_{\lambda_i} \langle\lambda_i|A\rangle\langle\lambda_i|$.
With $\langle\lambda_i|\lambda_j\rangle = \delta_{ij}$ and $\langle\lambda_i|\lambda_j\rangle = \langle\lambda_i|\lambda_j\rangle^*$
$\langle A|A\rangle = \sum_{\lambda_i} |\langle A|\lambda_i\rangle|^2 = P = 1.$

3.3 An Example Spin Operators

The goal is to write down the spin operators as 2×2 matrices.
Susskind paid attention to the subtle relation between operators and observables.

3.4 Constructing Spin Operators

The first goal is to construct operators to represent the components of spin:
$\sigma_x, \sigma_y,$ *and* σ_z .
Then, based on these spin operators a spin component in any direction is composed.
On page 76, Susskind applies the first three Principles as presented on pages 69 an 70, and I suppose these Principles to be confirmed by experiments. So, the result of an experiment is always one of the eigenvalues of the corresponding operator (Susskind page 71).
Dirac, in paragraph 10, used the expression: "It is reasonable……". No proof.

Exercise 3.2: About a spin operator

Prove that Eq. 3.16:

$$\begin{pmatrix}(\sigma_z)_{11} & (\sigma_z)_{12}\\ (\sigma_z)_{21} & (\sigma_z)_{22}\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

is the unique solution to Eqs. 3.14 and 3.15.

With Eq. 3.14 we find two equations:

$(\sigma_z)_{11} \cdot 1 + (\sigma_z)_{12} \cdot 0 = 1,$

and

$(\sigma_z)_{21} \cdot 1 + (\sigma_z)_{22} \cdot 0 = 0,$

giving the unique solutions

$(\sigma_z)_{11} = 1$ and $(\sigma_z)_{21} = 0.$

Eq. 3.15 leads to two equations:

$(\sigma_z)_{11} \cdot 0 + (\sigma_z)_{12} \cdot 1 = 0,$

and

$(\sigma_z)_{21} \cdot 0 + (\sigma_z)_{22} \cdot 1 = -1,$

giving the unique solutions

$(\sigma_z)_{12} = 0$ and $(\sigma_z)_{22} = -1.$

With the information derived in the foregoing sections, including the Principles, Susskind derived the expressions for σ_x, and σ_y .

3.5 A Common Misconception

Susskind dealt with the correspondence between operators and measurement. An example is given on page 82.

3.6 3-Vector Operators Revisited

The two notions of vectors are summarized:

- the vector in three-dimensional space,
- the state vector.

Operators have a lot in common with 3-vectors. Susskind, page 83, : "…, *it does no harm to think of them in that way*, …".

Then, the spin component of $\vec{\sigma}$ along the axis $\vec{n}$ is measured. The 3-vector character of $\vec{\sigma}$ is demonstrated. This leads to the general matrix representation of $\vec{\sigma}$, Eq.(3.23).

3.7 Reaping the Results

About real calculations.

Exercise 3.3: The eigenvectors and eigenvalues of σ_n

Calculate the eigenvectors and eigenvalues of σ_n.

$\sigma_n = \begin{pmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{pmatrix}$.

Hint: assume the eigenvector λ_1 has the form of

$\begin{pmatrix} cos\alpha \\ sin\alpha \end{pmatrix}$,

where α is an unknown parameter. Plug this vector into the eigenvalue equation and solve for α in terms of θ. Why did we use a single parameter α? Notice that our suggested column vector must have unit length.

A single parameter: there is one parameter in the matrix, θ. The column vector has unit length.

Consequently, $cos^2\alpha + sin^2\alpha = 1$.

I will use a more general approach.

The eigenvalues are found by the determinant (Chisholm and Morris) $\begin{vmatrix} cos\theta - \lambda & sin\theta \\ sin\theta & -cos - \lambda \end{vmatrix} = 0$, for nontrivial solutions.

Then,

$(cos\theta - \lambda) \cdot (-cos\theta - \lambda) - sin\theta \cdot sin\theta = -\cos^2\theta + \lambda^2 - \sin^2\theta = 0$.

Hence, the eigenvalues are: $\lambda = \pm 1$.

With these eigenvalues we can find the ratio of the components of the eigenvectors. First find the eigenvector $|\lambda_1\rangle$ and with eigenvalue $\lambda = 1$. We chose the components of the eigenvector to be β_1 and β_2. So,

$\begin{pmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{pmatrix}\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$:

$(cos\theta - 1)\beta_1 + \beta_2 sin\theta = 0$ and

$\beta_1 sin\theta - (cos\theta + 1)\beta_2 = 0$.

Hence $\frac{\beta_1}{\beta_2} = \frac{sin\theta}{1-cos\theta} = \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}$ and $\frac{\beta_1}{\beta_2} = \frac{1+cos\theta}{sin\theta} = \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}$.

With normalisation of the eigenvector

$\langle\lambda_1|\lambda_1\rangle = 1 = |\beta_1|^2 + |\beta_2|^2$,

we find $\beta_1 = cos\frac{\theta}{2}$ and $\beta_2 = sin\frac{\theta}{2}$.

Now the same procedure for the eigenvector $|\lambda_2\rangle$ with eigenvalue $\lambda = -1$. We chose for the components of the eigenvector: γ_1 and γ_2.

Similarly, we obtain

$(cos\theta + 1)\gamma_1 + \gamma_2 = 0$ and

$\gamma_1 sin\theta + (1 - cos\theta)\gamma_2 = 0$.

Hence $\frac{\gamma_1}{\gamma_2} = \frac{-sin\theta}{1+cos\theta} = -\frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}}$ and $\frac{\gamma_1}{\gamma_2} = \frac{cos\theta - 1}{sin\theta} = -\frac{sin\frac{\theta}{2}}{cos\frac{\theta}{2}}$.

With normalisation of the eigenvector, the components are: $\gamma_1 = -sin\frac{\theta}{2}$ and $\gamma_2 = cos\frac{\theta}{2}$.

Check: $\langle\lambda_1|\lambda_2\rangle = 0 \Longrightarrow |\lambda_2\rangle$ and $|\lambda_1\rangle$ are orthogonal.

Based on the above results, an experimental prediction is made. It is about the probability of observing $\sigma_n = +1$, page 87. Eqs. 3.24 and 3.25.

Remark:

On page 87 Susskind mentioned Principle 4 formulated on page 70, Eq. 3.11. The probability P is already presented in Eq. 2.2.

Susskind also introduced Eq. 3.26 : $\langle \boldsymbol{L} \rangle = \sum_i \lambda_i P(\lambda_i)$, the expectation value.

As we know, we can express a state vector $|\Psi\rangle$ in the eigenvectors $|\lambda_i\rangle$ as basis vectors and use a_i as probability amplitudes:

$|\Psi\rangle = \sum_i a_i\, |\lambda_i\rangle$.

Then $\langle \boldsymbol{L} \rangle = \langle \Psi | \boldsymbol{L} | \Psi \rangle = (\sum_i \langle \lambda_i | a_i^*) \boldsymbol{L} (\sum_i a_i | \lambda_i \rangle)$.

With the Kronecker Delta δ_{ij}, and $L|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$,

$\langle \boldsymbol{L} \rangle = \sum_i a_i^* a_i \lambda_i$,
where $a_i^* a_i$ is the probability $P(\lambda_i)$ to find a particle in state $|\lambda_i\rangle$. So,
$\langle \boldsymbol{L} \rangle = \sum_i \lambda_i P(\lambda_i)$.
On page 87 Susskind writes: *"What is the probability of observing $\sigma_n = +1$? "*
I consider this to be confusing. σ_n is an operator represented by a matrix; $+1$ is an eigenvalue of this matrix operator. So, equating the operator to the eigenvalue? On the other hand, considering observables and operators. I do understand the formulation.

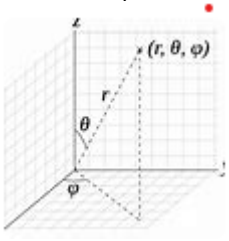
Exercise 3.4: The eigenvectors and eigenvalues of σ_n for spherical coordinates

Let $n_z = cos\theta$, $n_x = sin\theta cos\varphi$, and $n_y = sin\theta sin\varphi$. Angles θ and φ are defined according to the usual conventions of spherical coordinates (Fig. 3.2 Susskind). Compute the eigenvalues and eigenvectors of the matrix of Eq. 3.23.

Eq. 3.23:

$$\sigma_n = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}.$$

Then, with spherical coordinates



the determinant for the eigenvalues is

$$\begin{vmatrix} cos\theta - \lambda & sin\theta(cos\varphi - isin\varphi) \\ sin\theta(cos\varphi + isin\varphi) & -cos\theta - \lambda \end{vmatrix} = 0.$$

The equation for the eigenvalues is:

$$(cos\theta - \lambda) \cdot (-cos - \lambda) - sin\theta \cdot sin\theta \cdot (cos\varphi + isin\varphi) \cdot (cos\varphi - isin\varphi) =$$
$$= -\cos^2\theta + \lambda^2 - \sin^2\theta(\cos^2\varphi + \sin^2\varphi) = 0\,.$$

Again we find $\lambda = \pm 1$.

For the eigenvector $|\lambda_1\rangle$ and eigenvalue $\lambda = 1$, the components of the eigenvector are β_1 and β_2.

So,

$$\begin{pmatrix} cos\theta & sin\theta(cos\varphi - isin\varphi) \\ sin\theta(cos\varphi + isin\varphi) & -cos\theta \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

Similarly, to Exercise 3.3, we have

$\frac{\beta_1}{\beta_2} = (cos\varphi - isin\varphi)\frac{sin\theta}{1-cos\theta} = e^{-i\varphi}\frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}$, and $\frac{\beta_1}{\beta_2} = (cos\varphi - isin\varphi)\frac{1+cos\theta}{sin\theta} = e^{-i\varphi}\frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}$.

The components of the eigenvector $|\lambda_1\rangle$ are, using normalization,

$\beta_1 = cos\frac{\theta}{2}$ and $\beta_2 = sin\frac{\theta}{2}e^{i\varphi}$

Now, completely similar, for the eigenvector $|\lambda_2\rangle$ and the eigenvalue $\lambda = -1$, the components (γ_1, γ_2) of this eigenvector are, with normalization,

$\gamma_1 = -e^{-i\varphi}sin\frac{\theta}{2}$ and $\gamma_2 = cos\frac{\theta}{2}$.

Check: $|\lambda_2\rangle$ and $|\lambda_1\rangle$ are orthogonal.

Remark: $e^{-i\varphi}$ can be considered as a relative phase factor? Meaning?

Remark: Susskind alternately uses σ, $\vec{\sigma}$, $\hat{n}$ and n. A little bit confusing.

Exercise 3.5: The independence of the coordinate system and probabilities

Work out a more elaborate example involving the directions $\hat{n}$ and $\hat{m}$. In this setup the measuring equipment for the spin not only ends up in an arbitrary direction; it also starts out in a different arbitrary direction. Suppose the spin is prepared so that $\sigma_m = +1$.

Remark:

This is shorthand for the operator with eigenvalue $+1$.

I also assume that Susskind meant with the preparation $\sigma_m = +1$, the probability to find the spin in the "up position" is equal to 1. So, repeating measuring the spin in the $\hat{m}$ direction produces an eigen value of $+1$. The equipment is then rotated to the $\hat{n}$ direction and (the eigenvalue of) σ_n is measured. What is the probability that the result is $+1$? Note that $\sigma_m = \vec{\sigma} \cdot \hat{m}$ is similar to the convention we used for σ_n (Eq.3.22). This a lot of work indeed. However, we don't need $\sigma_m = \vec{\sigma} . \hat{m}$.

Since the physics does not depend on a coordinate system, we align for example the z-axis with $\hat{m}$ direction . Then we can use the same procedure as we did in Exercise 3.4. The probability $P(+1) = |\langle u|\beta_1\rangle|^2$. With β_1 as found in Exercise 3.4 and $\langle u| = (1,0)$: $P(+1) = cos^2\frac{\theta}{2}$. The angle θ given between $\hat{m}$ and $\hat{n}$.

Exercises 3.4 and 3.5 lead to the same eigenvalues and eigenvectors. This must be, since physics is invariant for a coordinate transformation.

3.8 The Spin-Polarization Principle

Any state of a single spin is an eigenvector of some component of the spin.

Proof:

Any spin state can be represented by: $|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$.

There exists some direction $\hat{n}$*, such that according to the principle*

$\vec{\sigma}.\hat{n}|A\rangle = |A\rangle$. ***(L3.8)***

With Eq. 3.23 and $|A\rangle$ *in column vector representation,* ***(L3.8)*** *gives*

$$\begin{pmatrix} n_z & (n_x - in_y) \\ (n_x + in_y) & -n_z \end{pmatrix}\begin{pmatrix}\alpha_u \\ \alpha_d\end{pmatrix} = \begin{pmatrix}\alpha_u \\ \alpha_d\end{pmatrix}.$$

Solving for α_u *and* α_d *we obtain*

$\alpha_u\left(n_x^2 + n_y^2 + n_z^2 - 1\right) = 0$

and

$\alpha_d\left(n_x^2 + n_y^2 + n_z^2 - 1\right) = 0.$

Excluding trivial solutions, α_u *and* α_d *to be zero, we have* $\left(n_x^2 + n_y^2 + n_z^2 - 1\right) = 0$

and that is to be expected. keeping in mind $\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$.

Consequently, $|A\rangle$ *is an eigenvector of* $\vec{\sigma}.\hat{n}|A\rangle$.

Since $|A\rangle$ *is an eigenvector of the operator* $\vec{\sigma}.\hat{n}$ *the expectation value of this operator is, with* ***(L3.8)*** *:*

$\langle\vec{\sigma}.\hat{n}\rangle = \langle A|\vec{\sigma}.\hat{n}|A\rangle = \langle A|A\rangle = 1,$

where the normalization of $|A\rangle$*,* $\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$ *is confirmed.*

End of Proof.

Question: I don't understand(page 91*): "On the other hand, the expectation value of the perpendicular components of* σ *(or* $\vec{\sigma}$ *) are zero in the state* $|A\rangle$*"*. Which are the perpendicular components of σ (or $\vec{\sigma}$)? σ_x, σ_y and σ_z?

Well, $\langle A|\sigma_z|A\rangle = \alpha_d^*\alpha_u + \alpha_u^*\alpha_d$.

So? I think the key is $\langle\sigma_n\rangle = \cos\theta$. The result of Exercises 3.4 and 3.5. Then perpendicular means $\theta = \pi/2$. For $\theta = 0$, we have the spin aligned along the z-axis and we find $\langle\sigma_x\rangle = 0$ and $\langle\sigma_y\rangle = 0$.

Susskind mentioned that the squares of the expectation values of all three components of the spin operator σ or $\vec{\sigma}$ sum up to 1:

With the Pauli Matrices(Eq. 3.20), the column vector representation for $|A\rangle$ and the definition of the expectation value you will find after some algebra:
$(\langle A|\sigma_x|A\rangle)^2 + (\langle A|\sigma_y|A\rangle)^2 + (\langle A|\sigma_z|A\rangle)^2 = (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)^2 = 1$. (Eq. 3.27).
***Note**: Dirac dealt with operators in chapter II. What intrigues me is his remark on page 28: "Thus the conjugate complex of the product of two linear operators equals the product of the conjugate complexes of the factors in reverse order. As simple examples of this result, it should be noted that, ξ and η are real, in general $\xi\eta$ is not real". Yet I could not find an example.*

Lecture 4. Time and Change.

In the preceding lectures the state-vectors have been analysed at a time, now it is time for "time and change". The time development operator, determinism in quantum mechanics and the Hamiltonian are introduced.

4.1 A Classical Reminder

In this introductory section Susskind made the important statement:*"… information is never lost."* The minus first Law.

4.2 Unitarity

A closed system is considered at time t in the quantum state $\Psi(t)$.

Remark: $\boldsymbol{U}$ as presented in Eq. 4.1 is the time-development operator for the system. I presume for every vector space or Landscape(Susskind) of vector spaces. The fact $\boldsymbol{U}^\dagger(t)\boldsymbol{U}(t)$ behaves like the unit operator $\boldsymbol{I}$ is related to the normalization of $|\Psi(t)\rangle$ stays normalized. On the pages 94 and 97 Susskind mentions Conservation of Distinction. I suppose this law is proved by measurement of observables, represented by the operator working on the state vector.

4.3 Determinism in Quantum Mechanics.

Susskind's Caveat: *Classical determinism allows us to predict the results of an experiment. The quantum evolution of states allows us to compute the probabilities of the outcomes of later experiments.*

4.4 A Closer Look at $U(t)$

Susskind starts with the requirements on $\boldsymbol{U}(t)$.
- A linear operator,
- Conservation of distinction.

Consequently, "*time evolution is unitary".*

Intermezzo A Universal Unitary 2-D Matrix with Elements represented by Real Numbers.

I assume the matrix elements of the 2-D unitary matrix U to be real numbers.

$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

In addition I consider U to be a Hermitian operator. Then,

$U = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$, and $U^{\dagger} = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$.

Next, let us find out about:

$U^{\dagger}U = I \Longrightarrow \begin{pmatrix} a & b \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Resulting into three equations:

- $a^2 + b^2 = 1$,
- $ab + bd = 0$,

and

- $b^2 + d^2 = 1$.

After some algebra, with $b \neq 0$, you obtain

$a = -d$,

and

$b = \pm\sqrt{1 - a^2}$.

Hence:

$b \neq 0$,

$U = \begin{pmatrix} a & \pm\sqrt{1-a^2} \\ \pm\sqrt{1-a^2} & -a \end{pmatrix}$,

and $b = 0$

$U = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Exercise 4.1: The unitary operator and the inner product of two vectors

Prove that if $\boldsymbol{U}$ is unitary, and if $|A\rangle$ and $|B\rangle$ are any two state-vectors, then the inner product of $\boldsymbol{U}|A\rangle$ and $\boldsymbol{U}|B\rangle$ is the same as the inner product of $|A\rangle$ and $|B\rangle$. One could call this the conservation of overlaps. It expresses the fact that the logical relation between states is preserved with time.

$\boldsymbol{U}$ is unitary and $\boldsymbol{U}^{\dagger}\boldsymbol{U} = \boldsymbol{I}$.

The inner product $\boldsymbol{U}|A\rangle$ and $\boldsymbol{U}|B\rangle$ is $\langle B\big|\boldsymbol{U}^{\dagger}\boldsymbol{U}\big|A\rangle = \langle B|\boldsymbol{I}|A\rangle = \langle B|A\rangle$.

4.5 The Hamiltonian

Susskind builds up changes of time by combining many infinitesimal time intervals ϵ. He mentioned that when ϵ is zero it should be obvious that in this case the time-evolution operator is merely the unit operator $\boldsymbol{I}$. It is obvious when considering Eq. 4.1 with $t = 0$: $|\Psi\rangle = \boldsymbol{I}|\Psi\rangle$.

Remark: On page 101 Susskind writes: *"…….also says that $\boldsymbol{H}$ is a Hermitian operator. This has great significance. We can now say that $\boldsymbol{H}$ is an observable, ……"*.

I suppose $\boldsymbol{H}$ to represent an observable.

On page 102 Susskind derives the generalized Schrödinger equation, Eq. 4.9:

$\frac{\partial|\Psi\rangle}{\partial t} = -i\boldsymbol{H}|\Psi\rangle$.

Remark: Susskind stated the following: "*We originally set things up so that the time variable is zero, but there was nothing special about $t = 0$. Had we chosen another time and done the*

same thing, we would have gotten the same result, namely, Eq. 4.9".
The equation $\boldsymbol{U}^{\dagger}\boldsymbol{U} = \boldsymbol{I}$:
Are we allowed to assume $\boldsymbol{U}$ to be Hermitian: $\boldsymbol{U}^{\dagger} = \boldsymbol{U}$?
$\boldsymbol{U}$ to be Hermitian gives four-time independent solutions: $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.
To demonstrate this, with $\boldsymbol{U}$ a 2×2 matrix with elements u_{ij}:
$\boldsymbol{U} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$.
Then,
$\boldsymbol{U}^{\dagger}\boldsymbol{U} = \boldsymbol{I} \rightarrow \begin{pmatrix} u_{11} & u_{21}^{*} \\ u_{12}^{*} & u_{22} \end{pmatrix}\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.
Assuming all the matrix elements to be real, we finally obtain: $\boldsymbol{U} = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.
With Eq. (4.1) we can set $\boldsymbol{U} = \boldsymbol{I}$ for $t = 0$.

***Note**: In the Feynman Lectures on Physics Vol. III chapter 8-4 "How states change with time" the above result for the generalized Schrödinger equation is derived a bit more elegantly. Feynman derived the Schrödinger equation at time t and did not use $t = 0$. Furthermore, instead of the state vector the equation is derived for the probability amplitude. In this way it is shown that the Schrödinger equation translates directly into the equation for the probability amplitude (and the wave function; see later). Feynman also wrote: "The Hamiltonian has one property that can be deduced right away, namely that $\boldsymbol{H}_{\boldsymbol{ij}}^{*} = \boldsymbol{H}_{\boldsymbol{ji}}$. This follows from the condition that the total probability that the system is in some state does not change".*

Question: There is something I do not understand with respect to the to the principle of unitarity (page 100) Eq. 4.5:
$\boldsymbol{U}^{\dagger}(\epsilon)\boldsymbol{U}(\epsilon) = \boldsymbol{I}$.
Plugging the expansions of $\boldsymbol{U}^{\dagger}(\epsilon) = \boldsymbol{I} + i\epsilon\boldsymbol{H}^{\dagger}$ and $\boldsymbol{U}(\epsilon) = \boldsymbol{I} - i\epsilon\boldsymbol{H}$ into the unitarity condition(principle) we obtain
$(\boldsymbol{I} + i\epsilon\boldsymbol{H}^{\dagger})(\boldsymbol{I} - i\epsilon\boldsymbol{H}) = \boldsymbol{I}$. Expanding to first order in ϵ, Susskind finds: $\boldsymbol{H}^{\dagger} = \boldsymbol{H}$**,** Eq.4.8.
When I expand the unitarity condition the result is: $i\epsilon\left(\boldsymbol{H}^{\dagger} - \boldsymbol{H}\right) = -\epsilon^{2}\boldsymbol{H}^{\dagger}\boldsymbol{H}$ or
$\boldsymbol{H}^{\dagger} - \boldsymbol{H} = i\epsilon\boldsymbol{H}^{\dagger}\boldsymbol{H}$ **.**
What does expand to first order in ϵ means in this respect, neglecting the right-hand side of $\boldsymbol{H}^{\dagger} - \boldsymbol{H} = i\epsilon\boldsymbol{H}^{\dagger}\boldsymbol{H}$? Well, I think this is correct. The term with ϵ can be neglected, a small quantity. So $\boldsymbol{H}^{\dagger} - \boldsymbol{H} = 0$.
The confusion arises when ϵ is applied for the deduction of the time-dependency of the state vector $|\Psi\rangle$. Then $\epsilon \rightarrow \Delta t \rightarrow dt$ in the limit and ϵ cannot be neglected.
Well, the limit $\epsilon \rightarrow 0$ must be considered.
Why did Susskind use the epsilontic approach? Is this in honour of Euler?[2]
Remark:
With the procedure at the bottom of page 101, Eq. 4.6 can be written as:
$\frac{\boldsymbol{U}(\epsilon) - \boldsymbol{U}(0)}{\epsilon} = -i\boldsymbol{H} \rightarrow \frac{d\boldsymbol{U}}{dt} = -i\boldsymbol{H}$, where $\boldsymbol{U}(0) = \boldsymbol{I}$,

[2] See: *A most elegant equation, Euler's formula & the beauty of mathematics* by David Stipp.

where use has been made of Eq.(4.1).
And with Eq. 4.7 :
$\frac{d\boldsymbol{U}^\dagger}{dt} = i\boldsymbol{H}^\dagger$. With $\boldsymbol{H}$ Hermitian $\frac{d\boldsymbol{U}}{dt} + \frac{d\boldsymbol{U}^\dagger}{dt} = 0$.
I am not so sure whether I am allowed to manipulate the time evolution operator in this way.

Remark: Susskind (page 101) writes below Eq.4.8: $\boldsymbol{H}^\dagger = \boldsymbol{H}$, "*This last equation expresses the unitary condition*". I prefer: this last equation is derived from the unitary condition Eq. 4.5. *Feynman concluded the Hermitian character of the Hamiltonian to be obtained from the total probability.*

4.6 What Ever Happened to $\hbar$?

Here the important analysis of dimensions is used.
Consequently, Eq.(4.9) is made consistent in terms of dimensions → Eq.(4.9).
Note: *Susskind paid some attention to: Why is Planck's constant so small? He is in good company. Schrödinger discussed this too in a slightly different manner: Why are atoms so small? This scale issue is also discussed by Feynman in Part I Chapter 19-2.*

4.7 Expectation Values

In this section the idea of the average or mean value is discussed.
On page 105 Susskind mentioned for the first time Dirac and the bra-ket notation in relation to the notation $\langle \boldsymbol{L} \rangle$. This is rather meagre since we used the bra and ket algebra for more than 80 pages already.
The mathematical formulation of an average is given in Eq.(4.11).
At the top of page 106 Susskind presented the basic hypothesis of statistical theory.
In Eq.(4.14), a quick rule to compute averages is presented
Page 108: the inner product of $\sigma_z|r\rangle$, Eq. 3.21, and $\langle r|$ gives $\langle \sigma_z \rangle = 0$:
$\langle \sigma_z \rangle = \langle r|\sigma_z|r\rangle = \frac{1}{\sqrt{2}}(\langle u| + \langle d|)\,\sigma_z \frac{1}{\sqrt{2}}(|u\rangle + |d\rangle) = \frac{1}{2}(\langle u| + \langle d|)(|u\rangle - |d\rangle) = 0$.

4.8 Ignoring the Phase-Factor

In lecture 4.8 Susskind explained the ignoring of the phase-factor.
This is illustrated by comparing two state vectors differing just a phase factor.
This can be found in Dirac on page 46.
See Exercise 3.4, page 21 of my notes: the relative phase factor.
Let us have a closer look at the phase factor in this exercise.
The two base vectors obtained in Exercise 3.4 are:

$$|\beta\rangle = \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix},$$

$$|\gamma\rangle = \begin{pmatrix}\gamma_1\\ \gamma_2\end{pmatrix} = \begin{pmatrix}-e^{-i\varphi}\sin\frac{\theta}{2}\\ \cos\frac{\theta}{2}\end{pmatrix}.$$

A state $|\psi\rangle$ can be expanded into the base vectors
$|\psi\rangle = \alpha_1\,|\beta\rangle + \alpha_2|\gamma\rangle$.

Next we try to find out about the observable $\boldsymbol{A}$ and determine its expectation value
$\langle\psi|\boldsymbol{A}|\psi\rangle = (\alpha_1^*\langle\beta| + \alpha_2^*\langle\gamma|)\boldsymbol{A}(\alpha_1\,|\beta\rangle + \alpha_2|\gamma\rangle)$.
To determine the role of the phase-factor, I use the column representation of the base vectors and matrix representation of the observable. A Hermitian matrix which I assumed to be diagonalized.
So, $\boldsymbol{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.
Then we have:

$$\langle\psi|\boldsymbol{A}|\psi\rangle = (\alpha_1^*\langle\beta| + \alpha_2^*\langle\gamma|)\boldsymbol{A}(\alpha_1\,|\beta\rangle + \alpha_2|\gamma\rangle) =$$

$$= \left[\alpha_1^*\left(\cos\tfrac{\theta}{2}\,, e^{-i\varphi}\sin\tfrac{\theta}{2}\right) + \alpha_2^*\left(-e^{i\varphi}\sin\tfrac{\theta}{2},\ \cos\tfrac{\theta}{2}\right)\right]\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\left[\alpha_1\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} + \alpha_2\begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}\right] =$$

$$= \left[\alpha_1^*\left(\cos\tfrac{\theta}{2}\,, e^{-i\varphi}\sin\tfrac{\theta}{2}\right) + \alpha_2^*\left(-e^{i\varphi}\sin\tfrac{\theta}{2}\,, \cos\tfrac{\theta}{2}\right)\right]\left[\alpha_1\begin{pmatrix} a\cos\frac{\theta}{2} \\ be^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} + \alpha_2\begin{pmatrix} -a\,e^{-i\varphi}\sin\frac{\theta}{2} \\ b\cos\frac{\theta}{2} \end{pmatrix}\right] = \alpha_1^*\alpha_1\, a\cos^2\tfrac{\theta}{2} + \alpha_1^*\alpha_1 b\sin^2\tfrac{\theta}{2} - \tfrac{\alpha_2^*\alpha_1 a}{2}e^{i\varphi}\sin\theta + \tfrac{\alpha_2^*\alpha_1 b}{2}e^{i\varphi}\sin\theta +$$

$$-\tfrac{\alpha_1^*\alpha_2 a}{2}e^{-i\varphi}\sin\theta + \tfrac{\alpha_1^*\alpha_2 b}{2}e^{-i\varphi}\sin\theta + \alpha_2^*\alpha_2 a\sin^2\tfrac{\theta}{2} + \alpha_2^*\alpha_2 b\,\cos^2\tfrac{\theta}{2}.$$

We do have some more information. From the eigen values and eigenvectors of A follows $a = b$.
Hence
$\langle\psi|\boldsymbol{A}|\psi\rangle = a(\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2)$.
So, we can ignore the phase factor?
Well, the above result comes as no surprise.
With $a = b$, we have
$\boldsymbol{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = a\boldsymbol{I}$.
Consequently,
$\langle\psi|\boldsymbol{A}|\psi\rangle = a\langle\psi|\boldsymbol{I}|\psi\rangle = a\langle\psi|\psi\rangle = a(\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2) = a$.
So what about
$\boldsymbol{A} = \begin{pmatrix} a & c \\ c^* & b \end{pmatrix}$?

$$\langle\psi|\boldsymbol{A}|\psi\rangle = (\alpha_1^*\langle\beta| + \alpha_2^*\langle\gamma|)\boldsymbol{A}(\alpha_1\,|\beta\rangle + \alpha_2|\gamma\rangle) =$$

$$= \left[\alpha_1^*\left(\cos\tfrac{\theta}{2}\,, e^{-i\varphi}\sin\tfrac{\theta}{2}\right) + \alpha_2^*\left(-e^{i\varphi}\sin\tfrac{\theta}{2}\,, \cos\tfrac{\theta}{2}\right)\right]\begin{pmatrix} a & c \\ c^* & b \end{pmatrix}\left[\alpha_1\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix} + \alpha_2\begin{pmatrix} -e^{-i\varphi}\sin\frac{\theta}{2} \\ \cos\frac{\theta}{2} \end{pmatrix}\right] =$$

$$= \left[\alpha_1^*\left(\cos\frac{\theta}{2}\,, e^{-i\varphi}\sin\frac{\theta}{2}\right) + \alpha_2^*\left(-e^{i\varphi}\sin\frac{\theta}{2}\,, \cos\frac{\theta}{2}\right)\right]\left[\alpha_1\begin{pmatrix} a\cos\frac{\theta}{2} + c\,e^{i\varphi}\sin\frac{\theta}{2} \\ c^*\cos\frac{\theta}{2} + be^{i\varphi}\sin\frac{\theta}{2}\end{pmatrix} + \right.$$

$$\left.\alpha_2\begin{pmatrix} -a\,e^{-i\varphi}\sin\frac{\theta}{2} + c\cos\frac{\theta}{2} \\ -c^*\,e^{-i\varphi}\sin\frac{\theta}{2} + \ b\cos\frac{\theta}{2}\end{pmatrix}\right] = \alpha_1^*\alpha_1\, a\cos^2\frac{\theta}{2} + \frac{\alpha_1^*\alpha_1 c}{2}e^{i\varphi}\sin\theta + \frac{\alpha_1^*\alpha_1 c^*}{2}e^{-i\varphi}\sin\theta +$$

$$+\alpha_1^*\alpha_1 b\sin^2\frac{\theta}{2} - \frac{\alpha_2^*\alpha_1 a}{2}e^{i\varphi}\sin\theta - \alpha_2^*\alpha_1 c\sin^2\frac{\theta}{2} + \alpha_2^*\alpha_1 c^*\cos^2\frac{\theta}{2} + \frac{\alpha_2^*\alpha_1 b}{2}e^{i\varphi}\sin\theta +$$

$$-\frac{\alpha_1^*\alpha_2 a}{2}e^{-i\varphi}\sin\theta + \alpha_1^*\alpha_2 c\cos^2\frac{\theta}{2} - \alpha_1^*\alpha_2 c^* e^{-2i\varphi}\sin^2\frac{\theta}{2} + \frac{\alpha_1^*\alpha_2 b}{2}e^{-i\varphi}\sin\theta +$$

$$+\alpha_2^*\alpha_2 a\sin^2\frac{\theta}{2} - \frac{\alpha_2^*\alpha_2 c}{2}e^{i\varphi}\sin\theta - \frac{\alpha_2^*\alpha_2 c^*}{2}e^{-i\varphi}\sin\theta + \alpha_2^*\alpha_2 b\cos^2\frac{\theta}{2}.$$

Now, it is not clear whether the phase shift has a role to play.

Well, may be using the matrix for σ_n in Eq.(3.23) we could find out about the phase factor. Furthermore, by averaging of φ over 2π , the phase factor disappears. Are we allowed to do the averaging?

Let us analyse the matrix $\sigma_n = \begin{pmatrix} \cos\theta & \sin\theta(\cos\varphi - i\sin\varphi) \\ \sin\theta(\cos\varphi + i\sin\varphi) & -\cos\theta\end{pmatrix}$.

So,

$a = \cos\theta,$

$b = -\cos\theta,$

$c = \sin\theta(\cos\varphi - i\sin\varphi) = e^{-i\varphi}\sin\theta,$

and

$c^* = \sin\theta(\cos\varphi + i\sin\varphi) = e^{i\varphi}\sin\theta.$

Then, we obtain:

- $-\frac{\alpha_1^*\alpha_1 c}{2}e^{i\varphi}\sin\theta = \frac{\alpha_1^*\alpha_1 e^{-i\varphi}\sin\theta}{2}e^{i\varphi}\sin\theta = \frac{1}{2}\alpha_1^*\alpha_1\sin^2\theta,$
- $-\frac{\alpha_1^*\alpha_1 c^*}{2}e^{-i\varphi}\sin\theta = \frac{\alpha_1^*\alpha_1 e^{-i\varphi}\sin\theta}{2}e^{i\varphi}\sin\theta = \frac{1}{2}\alpha_1^*\alpha_1\sin^2\theta,$
- $-\frac{\alpha_2^*\alpha_1 a}{2}e^{i\varphi}\sin\theta = -\frac{\alpha_2^*\alpha_1}{2}e^{i\varphi}\sin\theta\cos\theta,$
- $-\frac{\alpha_2^*\alpha_1 b}{2}e^{i\varphi}\sin\theta = -\frac{\alpha_2^*\alpha_1}{2}e^{i\varphi}\sin\theta\cos\theta\ ,$
- $-\frac{\alpha_1^*\alpha_2 a}{2}e^{-i\varphi}\sin\theta = -\frac{\alpha_1^*\alpha_2}{2}e^{-i\varphi}\sin\theta\cos\theta,$
- $-\alpha_1^*\alpha_2 c^* e^{-2i\varphi}\sin^2\frac{\theta}{2} = -\alpha_1^*\alpha_2 e^{i\varphi}\sin\theta\, e^{-2i\varphi}\sin^2\frac{\theta}{2} = -\alpha_1^*\alpha_2 e^{-i\varphi}\sin\theta\sin^2\frac{\theta}{2},$
- $-\frac{\alpha_1^*\alpha_2 b}{2}e^{-i\varphi}\sin\theta = -\frac{\alpha_1^*\alpha_2}{2}e^{-i\varphi}\sin\theta\cos\theta,$
- $-\frac{\alpha_2^*\alpha_2 c}{2}e^{i\varphi}\sin\theta = -\frac{\alpha_2^*\alpha_2}{2}e^{-i\varphi}\sin\theta\, e^{i\varphi}\sin\theta = -\frac{\alpha_2^*\alpha_2}{2}\sin^2\theta\ ,$
- $-\frac{\alpha_2^*\alpha_2 c^*}{2}e^{-i\varphi}\sin\theta = -\frac{\alpha_2^*\alpha_2}{2}e^{i\varphi}\sin\theta\, e^{-i\varphi}\sin\theta = -\frac{\alpha_2^*\alpha_2}{2}\sin^2\theta\ .$

Hence, there remain some expression with a phase factor:

- $-\frac{\alpha_2^*\alpha_1}{2}e^{i\varphi}\sin\theta\cos\theta - \frac{\alpha_2^*\alpha_1}{2}e^{i\varphi}\sin\theta\cos\theta = -\frac{\alpha_2^*\alpha_1}{2}e^{i\varphi}\sin 2\theta,$
- $-\alpha_1^*\alpha_2 e^{-i\varphi}\sin\theta\sin^2\frac{\theta}{2},$
- $-\frac{\alpha_1^*\alpha_2}{2}e^{-i\varphi}\sin\theta\cos\theta - \frac{\alpha_1^*\alpha_2}{2}e^{-i\varphi}\sin\theta\cos\theta = -\frac{\alpha_1^*\alpha_2}{2}e^{-i\varphi}\sin 2\theta.$

To be expected?

4.9 Connections to Classical Mechanics

Susskind: *"The average, or expectation value, of an observable is the closest thing in quantum mechanics to a classical value."*
Then, the time dependence of expectation values is discussed.
The concept of commutator is presented based on Eq.(4.16).
For any pair of operators, $\boldsymbol{L}$ and $\boldsymbol{M}$ page 111:
$[\boldsymbol{L},\boldsymbol{M}] = -[\mathbf{M},\mathbf{L}]$.
Well,
$\boldsymbol{ML} - \boldsymbol{LM} = [\boldsymbol{M},\boldsymbol{L}] \rightarrow -\boldsymbol{ML} + \boldsymbol{LM} = -[\boldsymbol{M},\boldsymbol{L}]$.

Exercise 4.2: $\boldsymbol{M}$ and $\boldsymbol{L}$ are both Hermitian, $i[\boldsymbol{M},\boldsymbol{L}]$ is Hermitian.

Prove that if $\boldsymbol{M}$ and $\boldsymbol{L}$ are both Hermitian, $i[\boldsymbol{M},\boldsymbol{L}]$ is also Hermitian. Note that the i is important. The commutator is, by itself, not Hermitian.
Proof:
Well, first not a real proof. A demonstration. We write the matrices $\boldsymbol{M}$ and $\boldsymbol{L}$ in components,
$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{12}^* & m_{22} \end{pmatrix}$ and $\boldsymbol{L} = \begin{pmatrix} l_{11} & l_{12} \\ l_{12}^* & l_{22} \end{pmatrix}$,
and calculate $i[\boldsymbol{M},\boldsymbol{L}] = i\boldsymbol{ML} - i\boldsymbol{LM}$**.** Having done this, calculate $(i[\boldsymbol{M},\boldsymbol{L}])^\dagger$. By comparing the components of $i[\boldsymbol{M},\boldsymbol{L}]$ and $(i[\boldsymbol{M},\boldsymbol{L}])^\dagger$, we see $i[\boldsymbol{M},\boldsymbol{L}]$ to be Hermitian.
A real proof is the following:
With $[\boldsymbol{M},\boldsymbol{L}] = \boldsymbol{ML} - \boldsymbol{LM}$ and $(\boldsymbol{ML})^\dagger = \boldsymbol{L}^\dagger\boldsymbol{M}^\dagger$ **,** we have
$[\boldsymbol{M},\boldsymbol{L}]^\dagger = (\boldsymbol{ML})^\dagger - (\boldsymbol{LM})^\dagger = \boldsymbol{L}^\dagger\boldsymbol{M}^\dagger - \boldsymbol{M}^\dagger\boldsymbol{L}^\dagger = \boldsymbol{LM} - \boldsymbol{ML}$**.**
In general, $\boldsymbol{LM} - \boldsymbol{ML} \neq \boldsymbol{ML} - \boldsymbol{LM}$.
Now $(i[\boldsymbol{M},\boldsymbol{L}])^\dagger = (i\boldsymbol{ML})^\dagger - (i\boldsymbol{LM})^\dagger = -i\boldsymbol{LM} + i\boldsymbol{ML} = i(\boldsymbol{ML} - \boldsymbol{LM}) = i[\boldsymbol{M},\boldsymbol{L}]$.
Hence $i[\boldsymbol{M},\boldsymbol{L}]$, is Hermitian.
End of Proof.

Then, Poisson brackets are introduced again. Susskind demonstrated the close relation between commutators and Poisson brackets.

Remark: In the text of page 113 Susskind mentioned G instead of H.

Exercise 4.3: The identification between commutators and Poisson brackets

Go back to the definition of Poisson brackets in *Volume I*(The Theoretical Minimum Series) and check that the identification in Eq. 4.21 is dimensionally consistent. Show that without the factor $\hbar$, it would not be.
Proof: the Poisson bracket is defined in Lecture 9 of Volume I, Eq. 9 page 172:
$\{F,G\} = \sum_i \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right)$,
where F and G are any two functions of phase space.
Given the dimension of $\hbar$ on page 103, kgm^2/s , we learn that the dimension of $\hbar$ corresponds with the dimension of $(q_i p_i)^{-1}$. Where q_i and p_i are the coordinates of phase space.

4.10 Conservation of Energy

Attention is paid to the meaning of conservation in quantum mechanics.
A quantity $\boldsymbol{Q}$ is conserved when, Eq.(4.19),
$\frac{d\boldsymbol{Q}}{dt} = 0$.

Remark: Susskind writes on page 115 that *"….if* $[\boldsymbol{H},\boldsymbol{Q}]=0$*, then* $[\boldsymbol{Q^2},\boldsymbol{H}]=0$*, or even more generally,* $[\boldsymbol{Q^n},\boldsymbol{H}]=0$*, for any* n*."*
We know $[\boldsymbol{H},\boldsymbol{Q}]=0$, so $\boldsymbol{HQ}-\boldsymbol{QH}=0$.
Multiply the $\boldsymbol{HQ}-\boldsymbol{QH}=0$ to the left with $\boldsymbol{Q}$ and add to the left hand side and the right hand side of the equality sign $\boldsymbol{HQ^2}$, we have
$\boldsymbol{QHQ}-\boldsymbol{Q^2H}+\boldsymbol{HQ^2}=\boldsymbol{HQ^2}$ or
$\boldsymbol{QHQ}-\boldsymbol{HQ^2}-\boldsymbol{Q^2H}+\boldsymbol{HQ^2}=0$
$(\boldsymbol{QH}-\boldsymbol{HQ})\boldsymbol{Q}-\boldsymbol{Q^2H}+\boldsymbol{HQ^2}=0\rightarrow\boldsymbol{Q^2H}-\boldsymbol{HQ^2}=0,$
so, $[\boldsymbol{Q^2},\boldsymbol{H}]=0$.
The same procedure and the method of induction can be used to prove $[\boldsymbol{Q^n},\boldsymbol{H}]=0$.

Proof by Induction
Assume $[\boldsymbol{Q^n},\boldsymbol{H}]=0$, to be true,
$\boldsymbol{HQ^n}-\boldsymbol{Q^nH}=0\Longrightarrow\boldsymbol{QHQ^n}-\boldsymbol{Q^{n+1}H}+\boldsymbol{HQ^{n+1}}=\boldsymbol{HQ^{n+1}}\Longrightarrow$
$\Longrightarrow\boldsymbol{QHQ^n}-\boldsymbol{HQ^{n+1}}-\boldsymbol{Q^{n+1}H}+\boldsymbol{HQ^{n+1}}=0\Longrightarrow(\boldsymbol{QH}-\boldsymbol{HQ})\boldsymbol{Q^n}+\boldsymbol{HQ^{n+1}}-\boldsymbol{Q^{n+1}H}=0.$
Then,
$\boldsymbol{HQ^{n+1}}-\boldsymbol{Q^{n+1}H}=0.$
Hence,
$[\boldsymbol{Q^{n+1}},\boldsymbol{H}]=0.$
End of Proof

The commutation of $\boldsymbol{Q}$ *with the Hamiltonian reflects the expectation values of all functions of* $\boldsymbol{Q}$ *to be conserved.*

4.11 Spin in a Magnetic Field

This section is about the application of the Hamiltonian for a single spin.
Susskind mentioned the time-dependence of an observable to be given by the commutator of the observable with the Hamiltonian, Eq.(4.19) shorthand for Eq.(4.18):
$\frac{d\boldsymbol{L}}{dt}=-\frac{i}{\hbar}[\boldsymbol{L},\boldsymbol{H}].$
Of importance is to find out about the Hamiltonian to describe the spin in a magnetic field $\vec{B}$.
The Hamiltonian, analogous to a classical spin in a magnetic field,
$H\propto\vec{\sigma}\cdot\vec{B}=\sigma_xB_x+\sigma_yB_y+\sigma_zB_z$
On page 117, Susskind applies Eq.(4.18) or (4.19).
With the expression for the Hamiltonian, Eq.(4.23) , Eqs.(4.25) are found.
Note: the "dot" above, e.g., $\sigma_x\Longrightarrow\dot{\sigma_x}$, is the fluxion notation of Newton.
So,
$\dot{\sigma_x}\equiv\frac{d\sigma_x}{dt}$.

Exercise 4.4 About the commutation relations of the Pauli matrices

Verify the commutation relations of Eqs. 4.26.

With the Pauli matrices summarized on page 347, I verify one of the commutation relations: $[\sigma_x, \sigma_y] = 2i\sigma_z$.

$[\sigma_x, \sigma_y] = \sigma_x\sigma_y - \sigma_y\sigma_x$, and, using, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $2i\sigma_z = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$.

Furthermore

$$\sigma_x\sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

and

$$\sigma_y\sigma_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

So,

$$\sigma_x\sigma_y - \sigma_y\sigma_x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix},$$

and consequently,

$[\sigma_x, \sigma_y] = 2i\sigma_z$.

Remark: Eq. 4.17 reads: $\frac{d}{dt}\langle \boldsymbol{L} \rangle = \frac{i}{\hbar}\langle [\boldsymbol{L}, \boldsymbol{H}] \rangle$ where $\boldsymbol{L}$ has no explicit time dependency. In Mahan you can find in the homework of Chapter I, Introduction, $\frac{d}{dt}\langle \boldsymbol{L} \rangle$ to be 0 in an eigenstate of $\boldsymbol{H}$ with discrete eigenvalues. Well, with Eq. 4.28, Eq. 4.17 can be written as:

$$\frac{d}{dt}\langle \boldsymbol{L} \rangle = \frac{i}{\hbar}\langle E_j | (\boldsymbol{HL} - \boldsymbol{LH}) | E_j \rangle = \frac{i}{\hbar}\langle E_j | E_j (\boldsymbol{L} - \boldsymbol{L}) E_j | E_j \rangle = \frac{iE_j}{\hbar}\langle E_j | (\boldsymbol{L} - \boldsymbol{L}) | E_j \rangle = 0.$$

Remember: eigenvalues of Hermitian operators are real; Susskind page 63. This is not all. When the eigenvectors form a complete set(page 66 and 67), any state vector can be expanded in the eigenvectors. Consequently, $\frac{d}{dt}\langle \boldsymbol{L} \rangle = 0$ for any state vector with $\boldsymbol{L}$ not explicitly dependent on time.

4.12 Solving the Schrödinger Equation

The time-dependent Schrödinger equation Eq.(4.10),

$$\hbar\frac{\partial|\Psi\rangle}{\partial t} = -i\boldsymbol{H}|\Psi\rangle ,$$

is the basis for this section.

The time-independent Schrödinger equation or eigenvalue equation is :

$H|E_j\rangle = E_j|E_j\rangle$, Eq. (4.28).

Exercise 4.5: The energy eigen values and eigenvectors of the spin Hamiltonian

Take any unit 3-vector $\hat{n} = (n_x, n_y, n_z)$, and form the operator $\boldsymbol{H} = \frac{\hbar\omega}{2}\,\vec{\sigma}\,.\hat{n}$.

Find the energy eigenvalues and eigenvectors by solving the time-independent Schrödinger equation.

Recall that Eq. 3.23 gives $\sigma_n = \vec{\sigma}\,.\hat{n}$, in matrix representation. A 2×2 matrix :

$$\sigma_n = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}.$$

There are two approaches for this exercise.

The first, 1), is use of the results of Exercise 3.4 and

the second, 2), to start with the time-independent Schrödinger equation.

ad 1) The eigenvalues of σ_n , found in Exercise 3.4, are ± 1. To obtain the energy eigenvalues of the operator $\boldsymbol{H}$ multiply ± 1 with $\frac{\hbar\omega}{2}$, and both energy eigenvalues are obtained. The eigenvectors are the same as found in Exercise 3.4:

$$|E_1\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ sin\frac{\theta}{2} \end{pmatrix} \text{ and } |E_2\rangle = \begin{pmatrix} -sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix}.$$

These column vectors can be rewritten in Cartesian coordinates with goniometric relations $sin\frac{\theta}{2} = \sqrt{(1 - cos\theta)/2}$, $cos\frac{\theta}{2} = \sqrt{(1 + cos\theta)/2}$, and $n_z = cos\theta$ (Lecture 3.7).

ad 2) $\vec{\sigma}\,.\hat{n} \Longrightarrow \sigma_n = \begin{pmatrix} n_z & n_x - in_y \\ n_x + in_y & -n_z \end{pmatrix}$.

With Eq. 4.28, the time-independent Schrödinger equation, we obtain with σ_n and $\boldsymbol{H}$

$\frac{\hbar\omega}{2}\sigma_n|E_j\rangle = E_j|E_j\rangle$ **,** $j = 1{,}2$.

The eigenvalues of the matrix are found with the determinant:

$$\begin{vmatrix} \frac{\hbar\omega}{2}n_z - E_j & \frac{\hbar\omega}{2}(n_x - in_y) \\ \frac{\hbar\omega}{2}(n_x + in_y) & -\frac{\hbar\omega}{2}n_z - E_j \end{vmatrix} = 0,$$

giving $-[\left(\frac{\hbar\omega}{2}n_z\right)^2 - E_j^2] - \left(\frac{\hbar\omega}{2}\right)^2 (n_x^2 - n_y^2) = 0.$

With $n_x^2 + n_y^2 + n_z^2 = 1,$

$E_j = \pm\frac{\hbar\omega}{2}.$

Remember, this shows once more the Spin Polarization Principle.

Now for the eigenvectors with column presentation

$|E_j\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} : \frac{\hbar\omega}{2}\sigma_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \pm\frac{\hbar\omega}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, we find two equations.

These are for $E_1 = +\frac{\hbar\omega}{2}$:

$n_z\alpha + (n_x - in_y)\beta = \alpha$ and

$(n_x + in_y)\alpha - n_z\beta = \beta$**.**

The preceding two equations give us with the ratio α/β , the eigenvector going with E_1:

$\alpha/\beta = -\frac{n_x - in_y}{n_z - 1}$, or $\alpha/\beta = \frac{n_z + 1}{n_x + in_y}$.

With normalization

$\alpha^*\alpha + \beta^*\beta = 1,$

and

$n_x^2 + n_y^2 + n_z^2 = 1,$

$\alpha = -\frac{1}{\sqrt{2}}\frac{n_x - in_y}{\sqrt{1 - n_z}}$,

$\beta = \frac{1}{\sqrt{2}}\sqrt{1 - n_z}$.

The eigenvector in column representation is $|E_1\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1 - n_z}}\begin{pmatrix} -n_x + in_y \\ 1 - n_z \end{pmatrix}$.

With the eigenvalue $E_2 = -\frac{\hbar\omega}{2}$, the two equations for components of the column representation of the eigenvector $|E_2\rangle$ are derived from $\frac{\hbar\omega}{2}\sigma_n \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = -\frac{\hbar\omega}{2}\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$:

$n_z\alpha + (n_x - in_y)\beta = -\alpha$ and

$(n_x + in_y)\alpha - n_z\beta = -\beta.$

These two equations give another ratio α/β for the eigenvector going with E_2:

$\alpha/\beta = -\frac{n_x - in_y}{n_z + 1}$, or $\alpha/\beta = -\frac{1 - n_z}{n_x + in_y}$.

With normalization

$\alpha^*\alpha + \beta^*\beta = 1$,

and

$n_x^2 + n_y^2 + n_z^2 = 1$,

$\alpha = -\frac{1}{\sqrt{2}}\frac{n_x - in_y}{\sqrt{1+n_z}}$,

$\beta = \frac{1}{\sqrt{2}}\sqrt{1+n_z}$

The eigenvector in column representation is $|E_2\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1+n_z}}\begin{pmatrix} -n_x + in_y \\ 1 + n_z \end{pmatrix}$.

After inspection we find $\langle E_1|E_2\rangle = 0$.

Notice: the complex factor in front of the square root can be written as a phase factor using $n_x^2 + n_y^2 + n_z^2 = 1$. The phase factor can be written as $e^{i2\phi}$, where ϕ is the angle between in_y and n_x in the two-dimensional complex sub-space.

Remark:

On page 121 Susskind writes: "…*the eigenvectors form an orthonormal basis…*". Do they? The Fundamental Theorem in Lecture 3.15 says: "*If …. are two unequal eigenvalues of a Hermitian operator, then the corresponding eigenvectors are orthogonal*". May be Susskind uses orthonormal and orthogonal alternately. When you express the eigenvectors in polar coordinates and neglect the phase vector $e^{-i\varphi}$ it is a bit more straightforward to show that the eigenvectors are normalized and orthonormal(See exercise 3.4).

On pages 121-124, Susskind derived the time dependent state-vector $|\Psi(t)\rangle$, Eq.(4.33).

4.13 Recipe for a Schrödinger Equation.

In this section the results of Lecture 4.12 have been summarized: a recipe to obtain the solution of the Schrödinger equation.

In Eq.4.33, the time-dependent solution of the Schrödinger equation is presented.

In the following exercise the recipe is applied.

The result is the possibility to predict the probabilities for each possible outcome of an experiment as a function of time.

In the exercise, at time t, a measurement is σ_y is made where the initial state is $|u\rangle$.

The final observable is σ_x.

Now, I like to draw tour attention to pages 12 and 13: Lecture 1.4 *Experiments Are Never Gentle "…..make an intermediate measurement(observation), and turn it(the apparatus) back to its original direction. Will a subsequent measurement along the z-axis confirm the original measurement? The answer is no".*

Exercise 4.6: Application of the Schrödinger Ket Recipe for a single spin

Carry out the Schrödinger Ket recipe for a single spin. The Hamiltonian is $\boldsymbol{H} = \frac{\omega\hbar}{2}\sigma_z$ and the final observable is σ_x. The initial state is given as $|u\rangle$ (the state in which $\sigma_z = +1$).
After time t, an experiment is done to measure σ_y. What are the possible outcomes and what are the probabilities for those outcomes?
Remark: before starting the recipe: I do not know what Susskind meant with final observable! May be it is somewhere hidden in the book.
The recipe: Step by Step.
1) Derive, look up, guess, borrow, or steal the Hamiltonian operator $\boldsymbol{H}$.
The Hamiltonian operator is given: $\boldsymbol{H} = \frac{\omega\hbar}{2}\sigma_z$.
2) Prepare the initial state $|\Psi(0)\rangle$.
The initial state is given: $|\Psi(0)\rangle = |u\rangle$, the state in which $\sigma_z = +1$.
3) Find the eigenvalues and eigenvectors of $\boldsymbol{H}$ by solving the time-independent Schrödinger equation, $\boldsymbol{H}|E_j\rangle = E_j|E_j\rangle$:

$$\begin{pmatrix} \frac{\hbar\omega}{2} & 0 \\ 0 & -\frac{\hbar\omega}{2} \end{pmatrix}|E_j\rangle = E_j|E_j\rangle.$$

The determinant for the eigenvalues is: $\begin{vmatrix} \frac{\hbar\omega}{2} - E_j & 0 \\ 0 & -\frac{\hbar\omega}{2} - E_j \end{vmatrix} = 0$,
then, $E_{1,2} = \pm\frac{\hbar\omega}{2}$.
The eigen vectors:

$$\begin{pmatrix} \frac{\hbar\omega}{2} & 0 \\ 0 & -\frac{\hbar\omega}{2} \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \pm\frac{\hbar\omega}{2}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

For $E_1 = \frac{\hbar\omega}{2}$:
$\alpha = \alpha$,
$-\beta = \beta$,
we have $\beta = 0$, and with normalization $\alpha = 1$.
Hence
$|E_1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$.
For $E_2 = -\frac{\hbar\omega}{2}$:
$\alpha = -\alpha$,
$-\beta = -\beta$,
we have $\alpha = 0$, and with normalization $\beta = 1$.
Hence
$|E_2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.
4) Use the initial state vector to be $|u\rangle$, along with the eigenvectors $|E_j\rangle$ from step 3 to calculate the initial coefficients $\alpha_j(0)$:
$\alpha_j(0) = \langle E_j|u\rangle$.
Then $\alpha_1(0) = \langle E_1|u\rangle = (1 \quad 0)\begin{pmatrix}1\\0\end{pmatrix} = 1$,
and
$\alpha_2(0) = \langle E_2|u\rangle = (0 \quad 1)\begin{pmatrix}1\\0\end{pmatrix} = 0$.
5) Rewrite $|\Psi(0)\rangle$ in terms of the eigenvectors $|E_j\rangle$ and the initial coefficients $\alpha_j(0)$:
$|\Psi(0)\rangle = \sum_j \alpha_j(0)|E_j\rangle \rightarrow |\Psi(0)\rangle = |E_1\rangle$.
6) In the above equation for $|\Psi(0)\rangle$, replace each $\alpha_j(0)$ with $\alpha_j(t)$ to capture time-dependence. As a result, $|\Psi(0)\rangle$ becomes $|\Psi(t)\rangle$: $|\Psi(t)\rangle = \sum_j \alpha_j(t)|E_j\rangle$.
7) Use Eq. 4.30: $\alpha_j(t) = \alpha_j(0)e^{-\frac{i}{\hbar}E_j t}$
replace each $\alpha_j(t)$ with $\alpha_j(0)e^{-\frac{i}{\hbar}E_j t}$ and the eigenvalue $E_1 = \frac{\hbar\omega}{2}$:

$|\Psi(t)\rangle = \sum_j \alpha_j(0) e^{-\frac{i}{\hbar}E_j t}|E_j\rangle = e^{-i\omega t/2}|E_1\rangle$.

Now, at time t, t_1 say, the experiment is done to measure σ_y. The operator for σ_y is the matrix $\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$.

So,

$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}|\Psi(t)\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}e^{-i\omega t/2}|E_1\rangle = \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}e^{-\frac{i\omega t}{2}}\begin{pmatrix}1\\0\end{pmatrix} = e^{-\frac{i\omega t}{2}}\begin{pmatrix}0\\i\end{pmatrix} = ie^{-\frac{i\omega t}{2}}\begin{pmatrix}0\\1\end{pmatrix} =$

$= e^{-\frac{i(\omega t-\pi)}{2}}\begin{pmatrix}0\\1\end{pmatrix} = |\Psi(t)\rangle$, for $t = t_1$,

or

$|\Psi(t_1)\rangle = e^{-\frac{i(\omega t_1-\pi)}{2}}|E_2\rangle$.

To predict the probabilities of the measurement σ_y, we need the eigenvectors of the matrix $\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$. The results of this exercise are presented in the Appendix and explained in detail in Lecture 3.4. I present here the results.

The eigenvalues are $\lambda_j = \pm 1$.

The eigenvectors which comply with normalization are

$$|\lambda_1\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{i}{\sqrt{2}}\end{pmatrix} \Longrightarrow |\lambda_1\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + i|E_2\rangle),$$

and

$$|\lambda_2\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-i}{\sqrt{2}}\end{pmatrix} \Longrightarrow |\lambda_2\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle - i|E_2\rangle).$$

Now, we need to calculate the two probabilities for both eigenvalues:

$P_{\lambda_1} = |\langle\Psi(t_1)|\lambda_1\rangle|^2$ and $P_{\lambda_2} = |\langle\Psi(t_1)|\lambda_2\rangle|^2$.

Plug the results for $t = t_1$ into $P_{\lambda_1} = |\langle\Psi(t_1)|\lambda_1\rangle|^2$, and you find after some algebra with complex numbers:

$P_{\lambda_1} = |\langle\Psi(t_1)|\lambda_1\rangle|^2 = \frac{1}{2}$,

consequently

$P_{\lambda_2} = |\langle\Psi(t_1)|\lambda_2\rangle|^2 = \frac{1}{2}$.

This is what we learned and should expect.

I suppose the final observable σ_x to be the observable at $t = t_2$ with $t_2 > t_1$. After the observation σ_y the system is in the state $|\lambda_1\rangle$ or $|\lambda_2\rangle$, which one we do not know. When we make the observation σ_x it is to be expected to find again equal probabilities.

So, with $\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ we find for the eigenvalues $\gamma_j = \pm 1$. The eigenvectors are

$$|\gamma_1\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix} \Longrightarrow |\gamma_1\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle + |E_2\rangle)$$

and

$$|\gamma_2\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}}\end{pmatrix} \Longrightarrow |\gamma_2\rangle = \frac{1}{\sqrt{2}}(|E_1\rangle - |E_2\rangle).$$

What about the probabilities?

$P_{\gamma_1} = |\langle\lambda_1|\gamma_1\rangle|^2$, under the condition $|\lambda_1\rangle$:

$P_{\gamma_1} = \frac{1}{2}(1-i)(1+i) = \frac{1}{2}$. The probability to be in $|\lambda_1\rangle$ is $\frac{1}{2}$. So the total probability to be in $|\gamma_1\rangle$ starting in $|\lambda_1\rangle$ is $P_{\gamma_1}\cdot\frac{1}{2} = \frac{1}{4}$.

Then we have: $P_{\gamma_1} = |\langle\lambda_2|\gamma_1\rangle|^2$, under the condition $|\lambda_2\rangle$. So the total probability to be in $|\gamma_1\rangle$ starting in $|\lambda_2\rangle$ is $P_{\gamma_1}\cdot\frac{1}{2} = \frac{1}{4}$.

Now the probability $P_{\gamma_2} = |\langle\lambda_1|\gamma_2\rangle|^2$ starting in $|\lambda_1\rangle$ is $P_{\gamma_2}\cdot\frac{1}{2} = \frac{1}{4}$.

The last probability to calculate is $P_{\gamma_2} = |\langle\lambda_2|\gamma_2\rangle|^2$ starting in $|\lambda_2\rangle$ is $P_{\gamma_2}\cdot\frac{1}{2} = \frac{1}{4}$.

Obviously, the total probability is equal 1.

4.14 Collapse

Lecture 4 concludes with the section about the collapse of the wave function of which the above exercise comprises an example. *See also Dirac page 36.*

An important remark by Susskind: *"Nevertheless, it is fair to say that between observations, the state of a system evolves in a perfectly definite way, according to the time-dependent Schrödinger equation."*

At the top of page 127, Susskind describes the phenomenon of collapse.

This leads to the conclusion for the need to describe the act of measurement(observation) by the laws of quantum mechanics.

Lecture 5. Uncertainty and Time Dependence.

5.1 Mathematical Interlude: Complete Sets of Commuting Variables

5.1.1 States that Depend On More Than One Measurable

A spin is an illustrative example of measurables, observables.

Susskind presented two examples of systems consisted of more than one spin. *There are multiple observables that are compatible, i.e., their values can be known simultaneously.*

Two examples are presented:

- the three position coordinates of a particle,
- a system composed of two physically independent spins.

Susskind discussed the latter example on the pages 130-133. It is about two different operators, specified at the bottom of page 131 and on the following pages of this section.

On page 133 Susskind writes: "*If an operator annihilates every member of a basis, it must also annihilate every vector in the vector space*". Why? Since any vector can be constructed by the members of that basis.

Important conclusion on the same page: *"……, the condition for two observables to be simultaneously measurable is that they commute."* , illustrated by Eq. (5.1):

$[\boldsymbol{L}, \boldsymbol{M}]|\lambda, \mu\rangle = 0$.

The complete set of commuting observables is introduced.

5.1.2 Wave functions

In this Lecture , the concept of wave function is introduced.

Susskind defined an orthonormal complete set of basis vectors, generally written as:

$|a, b, c, \ldots .\rangle$. I suppose, this general expression reflects the set:

$|a\rangle, |b\rangle, |c\rangle,$ " …. *and the eigenvalues are* $a, b, c,$ …. *of the complete set of commuting observables* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$ ".

Note: this sentence summarizes almost all the theory presented up to page 134. In addition, about $\boldsymbol{A}$, etc: observable or operator? It is subtle.

The arbitrary state vector Ψ is expanded in terms of the complete set of the basis vectors:

$|\Psi\rangle = \psi(a)|a\rangle + \psi(b)|b\rangle + \ \psi(c)|c\rangle + \cdots$.

Since, the basis is an orthonormal set, the inner product, Eq. (5.2)

$\langle a|\Psi\rangle = \psi(a)\langle a|a\rangle + \psi(b)\langle a|b\rangle + \psi(c)\langle a|c\rangle +, \ldots. = \psi(a).$

The coefficient $\psi(a)$ *is called the wave function.*

The probability $P(a)$ for the commuting observable $\boldsymbol{A}$ to have the eigenvalue a ,

$\boldsymbol{A}\,|\Psi\rangle = a\,|\Psi\rangle$,
$P(a) = \psi^*(a)\psi(a) = |\psi(a)|^2$, etc.

5.1.3 A Note About Terminology

In this part of the lecture, Susskind paid attention to the various notations used. As mentioned by Susskind, it is helpful "*to realize a wave function can represent a state vector".*

5.2 Measurement

In this Lecture the concept of measurement and commutation is discussed, page 137.
Then, Susskind returned to the single spin and a 2×2 Hermitian matrix, and explained this matrix to be represented by a sum of the three Pauli matrices and the unit matrix.

> **Exercise 5.1: A 2×2 Hermitian matrix can be written as a sum of four terms.**
> "*Any 2×2 Hermitian matrix $\boldsymbol{L}$ can be written as the sum of four terms,*
> $\boldsymbol{L} = a\sigma_x + b\sigma_y + c\sigma_z + dI$,
> *where $a, b, c,$ and d are real numbers."*
> Verify this claim.
> I's about the spin operators:
> $$\boldsymbol{L} = a\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + b\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + c\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + d\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}c+d & a-ib\\ a+ib & -c+d\end{pmatrix},$$
> An Hermitian matrix: *the diagonal elements being real and the other two being complex.*

5.3 The Uncertainty Principle

Uncertainty is one of the hallmarks of quantum mechanics, Susskind.
In this Lecture the general form of the uncertainty principle will be derived.

5.4 The Meaning of Uncertainty

Susskind introduces a new operator $\overline{\boldsymbol{A}}$ defined to be $\overline{\boldsymbol{A}} = \boldsymbol{A} - \langle \boldsymbol{A} \rangle$,
where $\langle \boldsymbol{A} \rangle$ is the expectation value, a real number.
The uncertainty in $\boldsymbol{A}$ is the so called standard deviation.
On top of page 141 Susskind writes: "*The eigenvectors of $\overline{\boldsymbol{A}}$ are the same as those of $\boldsymbol{A}$ and the eigenvalues are just shifted so that their average is zero as well*". Maybe it is just obvious but let us have a look.
For this we will use, as shown by Susskind, the identity operator $\boldsymbol{I}$, in order to transform the expectation value $\langle \boldsymbol{A} \rangle$ into an operator,
$\overline{\boldsymbol{A}} = \boldsymbol{A} - \langle \boldsymbol{A} \rangle \boldsymbol{I}$.
For a given state $|\Psi\rangle$:
$\overline{\boldsymbol{A}}|\Psi\rangle = (\boldsymbol{A} - \langle \boldsymbol{A} \rangle \boldsymbol{I})|\Psi\rangle = \boldsymbol{A}|\Psi\rangle - \langle \boldsymbol{A} \rangle \boldsymbol{I}|\Psi\rangle$.
Furthermore, we use the notation of Susskind for eigenvalues a of the operator $\boldsymbol{A}$. Then,
$\overline{\boldsymbol{A}}|\Psi\rangle = \boldsymbol{A}|\Psi\rangle - \langle \boldsymbol{A} \rangle \boldsymbol{I}|\Psi\rangle = a|\Psi\rangle - \langle \boldsymbol{A} \rangle|\Psi\rangle = (a - \langle \boldsymbol{A} \rangle)|\Psi\rangle$.
Page 141 : *"In other words, the eigenvalues of $\overline{\boldsymbol{A}}$ are $\bar{a} = a - \langle \boldsymbol{A} \rangle$."*
In addition Susskind writes: "*The probability distribution for $\overline{\boldsymbol{A}}$ is exactly the same as the distribution for $\boldsymbol{A}$ except that it is shifted so that the average value of $\overline{\boldsymbol{A}}$ is zero*".
Let's find out and use the notation of Susskind.
$\langle \Psi | \boldsymbol{A} | \Psi \rangle = \langle \boldsymbol{A} \rangle = \sum_a a\, P(a)$.
Now,

$\langle\Psi|\bar{\boldsymbol{A}}|\Psi\rangle = \langle\Psi|(\boldsymbol{A} - \langle\boldsymbol{A}\rangle\boldsymbol{I})|\Psi\rangle = \langle\Psi|\boldsymbol{A}|\Psi\rangle - \langle\Psi|\langle\boldsymbol{A}\rangle\boldsymbol{I}|\Psi\rangle = \sum_a a\,P(a) - \langle\boldsymbol{A}\rangle\langle\Psi|\Psi\rangle = \sum_a a\,P(a) - \langle\boldsymbol{A}\rangle = 0.$

The square of the uncertainty of $\boldsymbol{A}$, $(\Delta\boldsymbol{A})^2$ is presented in Eq. (5.3).

5.5 Cauchy-Schwarz Inequality

The basic mathematical inequality is the familiar triangle inequality.

See Figure 5.1.

5.6 The Triangle Inequality and the Cauchy-Schwarz Inequality

With help of a picture of the triangle, Figure 5.1, Susskind derived Eq.(5.9),

$2|X||Y| \geq |\langle X|Y\rangle + \langle Y|X\rangle|,$

where $|X|$ and $|Y|$ represent the length of the vectors, Figure 5.1 .

This equation represents the Cauchy-Schwarz inequality, leading to the uncertainty principle.

5.7 The General Uncertainty Principle

The uncertainty principal is presented in Eq.(5.13), derived from Eq.(5.9):

$\Delta\boldsymbol{A}\Delta\boldsymbol{B} \geq \frac{1}{2}|\langle\Psi|[\boldsymbol{A},\boldsymbol{B}]|\Psi\rangle.$

In the next exercise we pay attention to the derivation of Eq. (5.13).

There I assume $(\Delta\boldsymbol{A})^2 \equiv \Delta\boldsymbol{A}^2$, page 141.

Exercise 5.2 Some aspects of the Uncertainty Principal

1) Show that $(\Delta \boldsymbol{A})^2 = \langle(\overline{\boldsymbol{A}})^2\rangle$ and $(\Delta \boldsymbol{B})^2 = \langle(\overline{\boldsymbol{B}})^2\rangle$.
On page 141 Susskind: $(\Delta \boldsymbol{A})^2$ *may be written as* $(\Delta \boldsymbol{A})^2 = \langle\Psi|(\overline{\boldsymbol{A}})^2|\Psi\rangle$.
With the notation $\langle\Psi|(\overline{\boldsymbol{A}})^2|\Psi\rangle = \langle(\overline{\boldsymbol{A}})^2\rangle$, we have $(\Delta \boldsymbol{A})^2 = \langle(\overline{\boldsymbol{A}})^2\rangle$.
I am afraid this is just playing with symbols and no proof at all. We have to proof it, using Eq. (5.4):
$(\Delta \boldsymbol{A})^2 = \langle\Psi|(\overline{\boldsymbol{A}})^2|\Psi\rangle = \sum_i (a_i - \langle \boldsymbol{A}\rangle)^2 P(a_i)$, (L5.1)
here, for convenience, a notation slightly different from Susskind's is used.
a_i are the eigenvalues of operator $\boldsymbol{A}$ with eigenvectors $|a_i\rangle$.
Remark: Susskind introduced (L5.1) or Eq. (5.3) as a definition.
Now we expand the left-hand side of Eq. (L5.1), with $\overline{\boldsymbol{A}} = \boldsymbol{A} - \langle \boldsymbol{A}\rangle$:

$$\langle\Psi|(\overline{\boldsymbol{A}})^2|\Psi\rangle = \langle\Psi|(\boldsymbol{A} - \langle \boldsymbol{A}\rangle)^2|\Psi\rangle = \langle\Psi|\boldsymbol{A^2}|\Psi\rangle - 2\langle \boldsymbol{A}\rangle\langle\Psi|\boldsymbol{A}|\Psi\rangle + \langle\Psi\big|\langle \boldsymbol{A}\rangle^2\big|\Psi\rangle =$$

$$= \langle\Psi|\boldsymbol{A^2}|\Psi\rangle - 2\langle \boldsymbol{A}\rangle^2 + \langle \boldsymbol{A}\rangle^2\langle\Psi|\Psi\rangle = \langle\Psi|\boldsymbol{A^2}|\Psi\rangle - \langle \boldsymbol{A}\rangle^2. \quad \text{(L5.2)}$$

Keep in mind: in the operator mode $\langle \boldsymbol{A}\rangle$ should be read as $\langle \boldsymbol{A}\rangle \boldsymbol{I}$.
The right-hand side of Eq. (L5.1) is $\sum_i (a_i^2 - 2a_i\langle \boldsymbol{A}\rangle + \langle \boldsymbol{A}\rangle^2) P(a_i) =$

$$= \sum_i [a_i^2 P(a_i) - 2a_i\langle \boldsymbol{A}\rangle P(a_i) + \langle \boldsymbol{A}\rangle^2 P(a_i)]. \quad \text{(L5.3)}$$

With $\langle \boldsymbol{A}\rangle = \sum_i a_i \ P(a_i)$, and $\sum_i P(a_i) = 1$:
Eq. (L 5.3) results into:
$(\Delta \boldsymbol{A})^2 = \sum_i a_i^2 P(a_i) - \langle \boldsymbol{A}\rangle^2$. (L5.4)
The equality (L5.1) can be written with (L5.2) and (L5.4) as
$\langle\Psi|\boldsymbol{A^2}|\Psi\rangle - \langle \boldsymbol{A}\rangle^2 = \sum_i a_i^2 P(a_i) - \langle \boldsymbol{A}\rangle^2$. (L5.5)
Does (L.5.5) lead to a contradiction?
Now we investigate whether or not $\langle\Psi|\boldsymbol{A^2}|\Psi\rangle = \sum_i a_i^2 \ P(a_i)$.
To this end we expand $|\Psi\rangle$ into the basis of eigenvectors of the operator $\boldsymbol{A}$:
$|\Psi\rangle = \sum_i \alpha_i |a_i\rangle$.
Let $\boldsymbol{A}$ operate on $|\Psi\rangle$, then,
$\boldsymbol{A}|\Psi\rangle = \sum_i \alpha_i \boldsymbol{A}|a_i\rangle = \sum_i \alpha_i a_i |a_i\rangle$.
Now flip the ket $|\Psi\rangle$ into the bra $\langle\Psi|$, and we have $\langle\Psi| = \sum_j \alpha_j^* \langle a_j|$.
Let $\boldsymbol{A}$ operates on $\langle\Psi|$, this becomes $\langle\Psi\big|\boldsymbol{A} = \sum_j \alpha_j^* \langle a_j\big|\boldsymbol{A} = \sum_j \alpha_j^* \ a_j^* \langle a_j|$.
The inner product of $\langle\Psi|\boldsymbol{A}$, and $\boldsymbol{A}|\Psi\rangle$, results into:
$\langle\Psi|\boldsymbol{A^2}|\Psi\rangle = \sum_{\boldsymbol{ij}} \alpha_{\boldsymbol{j}}^* \alpha_j^* \alpha_i a_i \langle a_j\big|a_i\rangle = \sum_i \alpha_i^* \ \alpha_i^* a_i^2$, since $|a_i\rangle$, is an orthonormal base.
With $\alpha_i^* \alpha_i$, the probability P to find the observable in state i, we obtain the expression we are looking for:
$\langle\Psi|\boldsymbol{A^2}|\Psi\rangle = \sum_i a_i^2 \ P(a_i)$, and consequently,
$(\Delta \boldsymbol{A})^2 = \langle(\overline{\boldsymbol{A}})^2\rangle$.
For the operator $\boldsymbol{B}$ with eigenvalues b_i and eigenvectors $|b_i\rangle$, we derive similarly
$(\Delta \boldsymbol{B})^2 = \langle(\overline{\boldsymbol{B}})^2\rangle$.
2) Show that $[\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}] = [\boldsymbol{A}, \boldsymbol{B}]$. (L5.6)
$\overline{\boldsymbol{A}} = \boldsymbol{A} - \langle \boldsymbol{A}\rangle$, and $\overline{\boldsymbol{B}} = \boldsymbol{B} - \langle \boldsymbol{B}\rangle$. Substitute both expressions in Eq. (L5.6). After some algebra we obtain, using $\langle \boldsymbol{A}\rangle$ and $\langle \boldsymbol{B}\rangle$ to be real numbers:
$[\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}] = \boldsymbol{AB} - \boldsymbol{BA} = [\boldsymbol{A}, \boldsymbol{B}]$.
3) Using $(\Delta \boldsymbol{A})^2 = \langle(\overline{\boldsymbol{A}})^2\rangle$, $(\Delta \boldsymbol{B})^2 = \langle(\overline{\boldsymbol{B}})^2\rangle$, and $[\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}] = [\boldsymbol{A}, \boldsymbol{B}]$, show that:
$\Delta \boldsymbol{A} \Delta \boldsymbol{B} \geq \frac{1}{2} |\langle\Psi|[\boldsymbol{A}, \boldsymbol{B}]|\Psi\rangle|$.
With Eq. 5.12:
$2\sqrt{\langle \boldsymbol{A^2}\rangle\langle \boldsymbol{B^2}\rangle} \geq |\langle\Psi|[\boldsymbol{A}, \boldsymbol{B}]|\Psi\rangle|$, replacing $\boldsymbol{A}$ and $\boldsymbol{B}$ with $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ respectively, $\overline{\boldsymbol{A}}$ and $\overline{\boldsymbol{B}}$ both Hermitian operators, we obtain:
$2\sqrt{\langle(\overline{\boldsymbol{A}})^2\rangle\langle(\overline{\boldsymbol{B}})^2\rangle} \geq |\langle\Psi|[\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}]|\Psi\rangle|$. (L.5.7)
Now with the results of:
1) $(\Delta \boldsymbol{A})^2 = \langle(\overline{\boldsymbol{A}})^2\rangle$, and $(\Delta \boldsymbol{B})^2 = \langle(\overline{\boldsymbol{B}})^2\rangle$,
and
2) $[\overline{\boldsymbol{A}}, \overline{\boldsymbol{B}}] = [\boldsymbol{A}, \boldsymbol{B}]$,
substituted in Eq. (L5.7) $\rightarrow \Delta \boldsymbol{A} \Delta \boldsymbol{B} \geq \frac{1}{2} |\langle\Psi|[\boldsymbol{A}, \boldsymbol{B}]|\Psi\rangle|$.

Lecture 6. Combining Systems: Entanglement.

6.1 Mathematical Interlude: Tensor Products

6.1.1 Meet Alice and Bob

This lecture is about composite systems: A and B.
There are two spaces of states: S_A and S_B.
Susskind demonstrates a composite system by a quantum mechanical coin, A, and a quantum die, B.

6.1.2 Representing the Combined System

The combined system is constructed by a tensor product.
The symbol for this tensor product is: $\otimes$.
In Figure 6.1, Susskind presented the state labels for the combined system: S_{AB}. On the pages 153 and 154, Susskind explained the notation for the combined system S_{AB}.
At the end of this section, Susskind refers to an ongoing discussion, initiated by Einstein, about the differences between classical physics and quantum physics (Smolin,1).

6.2 Classical Correlation

With two coins Susskind explained the classical correlation. At the top of page 158 the statistical correlation is presented.

Exercise 6.1 Condition for zero correlation

Prove that if $P(a,b)$ factorizes, then the correlation between a and b is zero.
When $P(a,b)$ factorizes, a and b are independent. So, Eq.(6.3):
$P(a,b) = P_A(a)P_B(b)$.
The values of A's observations are σ_A and the B's observations are σ_B.
Then,
$\langle\sigma_A\sigma_B\rangle = \sum_{i,j} a_i b_j P(a_i, b_j) = \sum_{i,j} a_i P_A(a_i) b_j P_B(b_j) = \sum_i a_i P_A(a_i) \left[\sum_j b_j P_B(b_j)\right] =$
$= \sum_i a_i P_A(a_i) \langle\sigma_B\rangle = \langle\sigma_B\rangle \sum_i a_i P_A(a_i) = \langle\sigma_B\rangle\langle\sigma_A\rangle$.
So,
$\langle\sigma_A\rangle\langle\sigma_B\rangle \rightarrow \langle\sigma_A\sigma_B\rangle = \langle\sigma_A\rangle\langle\sigma_B\rangle$, the expectation value of the product is the product of the expectation values.
Hence,
$\langle\sigma_A\sigma_B\rangle - \langle\sigma_A\rangle\langle\sigma_B\rangle = 0$,
the correlation is zero.
Note, I did use Susskind's notation. However, an observation leads to the measurement of a_i, or b_j. So, instead of using $\langle\sigma_A\rangle$ and/or $\langle\sigma_B\rangle$, I should have used $\langle a\rangle = \sum_i a_i P_A(a_i)$,
$\langle b\rangle = \sum_j b_j P_B(b_j)$ and $\langle ab\rangle = \sum_{i,j} a_i b_j P(a_i, b_j)$.

To conclude this section, Susskind discussed the meaning of probability in classical physics.

6.3 Combining Quantum Systems

Now, Susskind repeated the experiment with A and B using spins instead of coins.
For a combined system, $|ab\rangle$ is used to represent the ket of this system.
Here, matrix elements are again introduced. Matrix elements are important parts of the quantum mechanical machinery. This explained on page 161.

6.4 Two spins

The notations for the two-spin system is presented.
The space of states is a tensor product.
The basis states are presented, top page 163.

6.5 Product States.

The product state is the simplest type of state for the composite system.
Susskind based the product state on the two states $|A\rangle$ and $|B\rangle$.
I do not use the Susskind notation: $|A\} = \alpha_u|u\} + \alpha_d|d\}$.
I use the bra and ket notation of Dirac.
Reminder normalization:
$\langle A|A\rangle = (\langle u|\alpha_u^* + \langle d|\alpha_d^*)(\alpha_u|u\rangle + \alpha_d|d\rangle) = 1,$
with
$\alpha_u^*\alpha_d\langle u|d\rangle = 0, \alpha_d^*\alpha_u\langle d|u\rangle = 0$.
The state of B is:
$\beta_u|u\rangle + \beta_d|d\rangle.$
The product state is presented at the top of page 164, Eq. (6.5).

Exercise 6.2: About normalization of a product state

Show that if the two normalization conditions of Eqs. 6.4 are satisfied, then the state-vector of Eq. 6.5 is automatically normalized as well. In other words, show that for this $|product\ state\rangle$, normalizing the overall state-vector does not put any additional constraints on the α's and β's.
A's state:
$\alpha_u|u\rangle + \alpha_d|d\rangle,$
and the state of B:
$\beta_u|u\rangle + \beta_d|d\rangle.$
The normalization condition for both states gives, the usual procedure with inner products:
$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$, and $\beta_u^*\beta_u + \beta_d^*\beta_d = 1$, Eqs.(6.4).
The combined system:
$|product\ state\rangle = [\ \alpha_u|u\rangle + \alpha_d|d\rangle] \otimes [\ \beta_u|u\rangle + \beta_d|d\rangle].$
With the composite notation, the product state-vector $|\Psi\rangle$ becomes, Eq.(6.5):
$|\Psi\rangle = \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle.$
For this product state-vector to be normalized, the condition is: $\langle\Psi|\Psi\rangle = 1.$
The bra $\langle\Psi|$ results from the complex conjugate of Eq.(6.5):
$\langle\Psi| = \langle uu|\alpha_u^*\beta_u^* + \langle ud|\alpha_u^*\beta_d^* + \langle du|\alpha_d^*\beta_u^* + \langle dd|\alpha_d^*\beta_d^*.$ **(L6.1)**
Now we take the inner product of $\langle\Psi|$ and $|\Psi\rangle$, thereby using $\langle ab|a'b'\rangle = \delta_{aa'}\delta_{bb'}$, defined at page 161.
The Kronecker delta is zero unless $a = a'$ and $b = b'$.
For example: $\langle uu|dd\rangle = 0.$
So, with $|\Psi\rangle$, given in Eq.(6.5) and $\langle\Psi|$, given in **(L6.1)** :
$\langle\Psi|\Psi\rangle = (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)(\beta_u^*\beta_u + \beta_d^*\beta_d) = 1.$
The normalization creates no additional constraints on the α's and β's.

6.6 Counting Parameters for the Product State

In this Lecture, Susskind demonstrates an important technique: counting parameters to illustrate consistency. Do compare this with dimension analysis.

6.7 Entangled States.

As mentioned in Lecture 6.5: *"… most of the state vectors in the product space are not product states."*
At the top of page 166: *"… we only have one normalization condition."*
Entanglement is introduced.

Exercise 6.3: The maximal entangled state-singlet

Prove that the state $|sing\rangle$ cannot be written as a product state.
$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$.
Eq. (6.5), page 164, the product state:
$|\Psi\rangle = \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$,
represents the product state. Compare Eq. (6.5) with the expression for $|sing\rangle$:
$|sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$.
Then,
$\alpha_u\beta_u|uu\rangle = 0$,
$\Longrightarrow \alpha_u$ and/or β_u are/is zero.
Consequently,
$\alpha_u\beta_d|ud\rangle = 0$, or $\alpha_d\beta_u|du\rangle = 0$.
Also,
$\alpha_d\beta_d|dd\rangle = 0$,
$\Longrightarrow \alpha_d$ and/or β_d are/is zero.
Consequently,
$\alpha_u\beta_d|ud\rangle = 0$, or $\alpha_d\beta_u|du\rangle = 0$.
Hence, the state $|sing\rangle$ cannot be written as a product state.

Remark: Susskind writes on page 160: "*We will make frequently use of the notation $|ab\rangle$ to label a single basis vector of the combined system*". On page 162 he writes: "*Let's work in a basis in which the z components of both spins are specified. The basis vectors are: $|uu\rangle, |ud\rangle, |du\rangle, |dd\rangle$,….*". On page 164 and page 165 he writes: "*I'll mention here that tensor products and product states are two different things, despite their similar-sounding names. (Footnote: Sometimes, we'll use the term tensor product space, or just product space, instead of tensor product). A tensor product is a vector space for studying composite systems. A product state is a state-vector. It's one of the many state-vectors that inhabit a product space. As we will see, most of the state-vectors in the product space are not product states*".
On page 169 Susskind writes: "*Now let's consider how the operators should be defined when acting on the tensor product states $|uu\rangle, |ud\rangle, |du$, and $|dd\rangle$*". Here Susskind uses the expression: tensor product states. These are the single basis vectors for the combined system. The word "product" belongs to tensor.
Page 167: Susskind spent a few words on the mystery of entanglement.

6.8 Alice and Bob's Observables.

In this Lecture, Susskind presents the results of the spin operators acting on the system: a composite space of states. See Eqs. (6.6) and (6.7).
In this Lecture the observables of spin measurements are presented. Furthermore, the convention how to operate on the product states. Attention is paid to the tensor product space. Compare Eqs. (6.9) and (6.10).

At the bottom of page 170, Susskind makes the remark: *"If we were being pedantic, we would insist on writing the tensor product version of σ_z and τ_x as $\sigma_z \otimes I$ and $I \otimes \tau_x$……."*.
Pedantic? It depends. The alternative is to remember, Eq.(6.9),
$\sigma_z|du\rangle = -|du\rangle$,
σ_z operates on d, and the result is $-|du\rangle$.
To show the elegance of Tensor Calculus, see the example below.

Example of the elegance of Tensor Calculus

Using the tensor product for $I \otimes \tau_x$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and e.g.,

$$|ud\rangle = |u\rangle \otimes |d\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

we have

$$(I \otimes \tau_x)(|u\rangle \otimes |d\rangle) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |u\rangle \otimes |u\rangle = |uu\rangle.$$

See Eq.(6.8).
This seems to be a lot of work. On the other hand, studying, e.g., Quantum Mechanics, Special Relativity end Classical Field Theory(Susskind Volume III) , applying tensor calculus is not a real burden.
Next the product of operators
$\sigma_x\tau_x$,
operating on $|uu\rangle$.

$$\sigma_x\tau_x \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

$$|uu\rangle \to \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Now

$$\sigma_x\tau_x|uu\rangle \Longrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \Longrightarrow |dd\rangle.$$

From this result we conclude σ_x operates on the left part of $|uu\rangle$, and τ_x on the right part of $|uu\rangle$.
Note; in Lecture 7.1.2 Susskind showed the elegance of tensor products with additional examples.

Exercise 6.4: operate the spin matrix operators on the up and down column vectors.

1)Use the matrix forms of σ_x, σ_y and σ_z and the column vectors for $|u\rangle$ and $|d\rangle$ to verify Eqs. (6.6). Note: I don't use Susskind's notation $|\}$.
2)Then, use Eqs. (6.6) and (6.7) to write the equations that were left out of Eqs. (6.8). Use the appendix to check your answers.
ad 1) Let us take for example σ_y and $|u\rangle$:
$\sigma_y|u\rangle = i|d\rangle$. We write this expression in matrix and column vector representation:
$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
The left-hand side becomes $\begin{pmatrix} 0 \\ i \end{pmatrix}$ and this equals the right-hand side.
ad 2) Let us for example write out the equation $\sigma_z|ud\rangle$.
The convention, with Eq. 6.6:
$\sigma_z|u\rangle = |u\rangle$,
we have
$\sigma_z|ud\rangle = |ud\rangle$.

Exercise 6.5: spin operators and the product state.

1) Prove the following theorem:
When one of A's (σ) or B's (τ) spin operators acts on a product state, the result is still a product state(not entangled). See also Exercise 6.2.
2) Show that in a product state(not entangled), the expectation value of any component of $\vec{\sigma}$ or $\vec{\tau}$ is exactly the same as it would be in the individual single-spin states.
ad 1) So let's take σ_x and the product state given in Eq.(6.5):
$\sigma_x|\Psi\rangle = \sigma_x(\alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle)$, **(L6.2)**
we know now how σ_x operates on the basis vectors. So Eq. **(L6.2)** becomes:
$\sigma_x|\Psi\rangle = \alpha_u\beta_{\boldsymbol{u}}|du\rangle + \alpha_u\beta_d|dd\rangle + \alpha_d\beta_u|uu\rangle + \alpha_d\beta_{\boldsymbol{d}}|ud\rangle$,
this can be written as:
$(\alpha_u|d\rangle + \alpha_d|u\rangle) \otimes (\beta_u|u\rangle + \beta_d|d\rangle)$, representing a product state, the Theorem.
ad 2) The expectation value of any component of $\vec{\sigma}$ and $\vec{\tau}$ is the same as in the single-spin state.
Remember: we work in a basis in which the z components of both spins are specified.
For the product state $|\Psi\rangle$, the expectation value:
$\langle\sigma_x\rangle = \langle\Psi|\sigma_x|\Psi\rangle$.
With **(L6.2)** and Eq. 6.5, the expectation value, using $\langle ab|a'b'\rangle = \delta_{aa'}\delta_{bb'}$:
$\langle\sigma_x\rangle = (\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)(\beta_u^*\beta_u + \beta_d^*\beta_d)$.
Use $\beta_u^*\beta_u + \beta_d^*\beta_d = 1$, the normalisation condition for a product state Eq.(6.4), we obtain:
$\langle\sigma_x\rangle = (\alpha_u^*\alpha_d + \alpha_d^*\alpha_u)$
The state vectors $|ab\rangle$, are taken orthonormal. This is expressed by the Kronecker delta symbol $\langle ab|a'b'\rangle = \delta_{aa'}\delta_{bb'}$.
For example $\langle ud|uu\rangle = \delta_{ud}\delta_{uu} = 0$ and $\langle ud|ud\rangle = \delta_{uu}\delta_{dd} = 1$.
The single-spin state:
$|A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$ gives for the expectation value $\langle\sigma_x\rangle = \langle A|\sigma_x|A\rangle$:
$\langle\sigma_x\rangle = \alpha_u^*\alpha_d + \alpha_d^*\alpha_u$. The same as for the product state.
Similarly, we find for $\langle\sigma_y\rangle$:
$\langle\sigma_y\rangle = \langle\Psi|\sigma_y|\Psi\rangle = i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)$, the same as for the single-spin state.
Finally:
$\langle\sigma_z\rangle = \langle\Psi|\sigma_z|\Psi\rangle = -\alpha_u^*\alpha_d + \alpha_d^*\alpha_u$, the same as for the single-spin state.
For the single-spin state we proved in **Lecture 3:** Eq. (3.27). There Susskind writes: "*Moreover, this is true for any state*". We have proven in this Exercise, Eq (3.27) to be true for the two-spin product state.

For completeness, Eqs. (3.27) and (6.11):

$$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)^2 = 1.$$

Now we must prove this for $\vec{\tau}$. It will be of no surprise that we find the same results for the same product state $|\Psi\rangle$ and the single-spin state $|B\rangle = \beta_u|u\rangle + \beta_d|d\rangle$.
Furthermore, the normalisation condition we use in this case reads: $\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$ and again the Kronecker delta for the two-spin state.
Another example: $\langle dd|ud\rangle = \delta_{dd}\delta_{ud} = 0$, and $\langle du|du\rangle = \delta_{dd}\delta_{uu} = 1$.
Susskind presented on pages 173 and 174 the expectation values for σ_x and the entangled state $|sing\rangle$.

6.9 Composite Observables

Remark: On page 176 Susskind writes: "*But for Alice and Bob, it is easy to see that every component of* $\sigma(\equiv \vec{\sigma})$ *commutes with every component of* $\tau(\equiv \vec{\tau}\,)$*"*. Every component? Well, see Exercise (6.6) below: $[\sigma_x, \tau_y] = 2i\sigma_z$. However, should we read this sentence on commutation as, e.g., $[\sigma_x, \tau_x] = [\sigma_z, \tau_z] = [\sigma_y, \tau_y] = 0$? I think so.
In this Lecture Susskind deals with the composite operators and the related observables. As an example, the state $|sing\rangle$ is used.

Exercise 6.6 The expectation value of spin operators for a singlet state

1)Assume Charlie has prepared the two spins in the singlet state. This time, Bob measures τ_y and Alice measures σ_x. What is the expectation value of $\sigma_x\tau_y$?
2) What does this say about the correlation between the two measurements?
ad 1) $\sigma_x\tau_y|sing\rangle = \sigma_x\tau_y\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle)$,
where σ_x operates on the left-hand side of the basis vector and τ_y operates on the right-hand side of the basis vector. So,
$\sigma_x\tau_y|sing\rangle = \sigma_x\frac{1}{\sqrt{2}}(-i|uu\rangle - i|dd\rangle) = -\frac{i}{\sqrt{2}}(|du\rangle + |ud\rangle)$.
The last term between brackets is not a singlet and $|sing\rangle$ is not an eigenvector of $\sigma_x\tau_y$.
The expectation value of $\sigma_x\tau_y$:
$\langle\sigma_x\tau_y\rangle = \langle sing|\sigma_x\tau_y|sing\rangle = -\frac{i}{\sqrt{2}}\langle sing|(|du\rangle + |ud\rangle) =$
$\frac{-i}{2}(\langle ud| - \langle du|)(|du\rangle + |ud\rangle) = \frac{-i}{2}(\langle ud|ud\rangle + \langle ud|du\rangle - \langle du|ud\rangle - \langle du|du\rangle) = \frac{-i}{2}(1 + 0 - 0 - 1) = 0.$
Use has been made of the Kronecker delta for the combined state.
The expectation value is zero.
ad 2) The correlation $\langle\sigma_x\tau_y\rangle - \langle\sigma_x\rangle\langle\tau_y\rangle$ is found form the above result for $\langle\sigma_x\tau_y\rangle$ and the expectation values $\langle\sigma_x\rangle$ and $\langle\tau_y\rangle$ as calculated by Susskind on page 173 and 174.
Then,
$\langle\sigma_x\tau_y\rangle - \langle\sigma_x\rangle\langle\tau_y\rangle = 0 - 0 = 0 \rightarrow$ no correlation.
Do σ_x and τ_y commute: $[\sigma_x, \tau_y] = 0$ zero?
With the Pauli matrices we obtain:
$[\sigma_x, \tau_y] = \sigma_x\tau_y - \tau_y\sigma_x = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 2i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = 2i\sigma_z \neq 0.$
These components of $\sigma(\equiv \vec{\sigma})$ and $\tau(\equiv \vec{\tau}\,)$ do not commute . Nothing new: see page 118 and Eq. (4.26). This defines the expression "*every component*" of page 176 and explains the question mark in the above remark.
However, I think the remark of Susskind means: $[\sigma_x, \tau_x] = [\sigma_z, \tau_z] = [\sigma_y, \tau_y] = 0$.

A reminder for the subsequent exercises:
$\langle uu|uu\rangle = 1,$
$\langle ud|ud\rangle = 1,$

$\langle du|du\rangle = 1,$
$\langle dd|dd\rangle = 1\,.$

Exercise 6.7: The expectation value of spin operators for a triplet state

Next, Charlie prepares the spins in a different state, called $|T_1\rangle$, where
$|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$.
In these examples, T stands for *triplet.* These triplet states are completely different from the states in the coin and die examples.
What are the expectation values for $\sigma_z\tau_z$, $\sigma_x\tau_x$, and $\sigma_y\tau_y$?
$\langle\sigma_z\tau_z\rangle = \langle T_1|\sigma_z\tau_z|T_1\rangle$. With Eqs. 6.8:
$\langle\sigma_z\tau_z\rangle = -\frac{1}{2}(\langle ud|+\langle du|)(|ud\rangle + |du\rangle)$,
where use has been made of Table 1 of the Appendix.
Applying the Kronecker delta for the combined state: $\langle\sigma_z\tau_z\rangle = -1$.
This can be obtained in a slightly different way.
$\sigma_z\tau_z|T_1\rangle = -\frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle) = -|T_1\rangle$. We see $|T_1\rangle$ to be an eigenvector of the operator $\sigma_z\tau_z$ with eigenvalue -1. The expectation value can then be obtained in the following way: $\langle\sigma_z\tau_z\rangle = \langle T_1|\sigma_z\tau_z|T_1\rangle = -\langle T_1|T_1\rangle = -1$.
Similarly we have for $\langle\sigma_x\tau_x\rangle = 1$, where $\sigma_x\tau_x|T_1\rangle = |T_1\rangle$. We see $|T_1\rangle$ to be an eigenvector of the operator $\sigma_x\tau_x$, with eigenvalue $+1$.
And $\sigma_y\tau_y|T_1\rangle = |T_1\rangle$.
The triplet $|T_1\rangle$ is an eigenvector of the operator $\sigma_y\tau_y$, with eigenvalue $+1$.
The expectation value $\langle\sigma_y\tau_y\rangle = 1$.
$\langle T_1|\sigma_x|T_1\rangle = \frac{1}{2}(\langle ud|+\langle du|)\sigma_x(|ud\rangle + |du\rangle) = \frac{1}{2}(\langle ud|+\langle du|)(|ud\rangle - |du\rangle) =$
$= 1 + 0 + 0 - 1 = 0$,
where use has been made of the Kronecker delta for combined states.
Similarly
$\langle\sigma_y\rangle = 0$, and $\langle\sigma_z\rangle = 0$.
Then, e.g.,
$\langle\sigma_y\tau_y\rangle - \langle\sigma_y\rangle\langle\tau_y\rangle = 1$
Consequently, we have *perfect* correlation when maximally correlated.

Exercise 6.8 The expectation value of spin operators for the other two triplet states

Do the same as in Exercise 6.7 for the other two entangled triplet states,
$|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$ and $|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle)$, and interpret.
$|T_2\rangle$:
$\langle\sigma_z\tau_z\rangle$, with Eq.(6.8) or Appendix Table 1:
$\sigma_z\tau_z|T_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle) = |T_2\rangle.$
So,
$|T_2\rangle$, is an eigenvector of the operator $\sigma_z\tau_z$ with eigenvalue $+1$.
Consequently,
$\langle\sigma_z\tau_z\rangle = \langle T_2|\sigma_z\tau_z|T_2\rangle = \langle T_2|T_2\rangle = 1.$
$\langle\sigma_x\tau_x\rangle$, with Eq.(6.8):
$\sigma_x\tau_x|T_2\rangle = \frac{1}{\sqrt{2}}(|dd\rangle + |uu\rangle) = |T_2\rangle.$
So,
$|T_2\rangle$, is an eigenvector of the operator $\sigma_x\tau_x$ with eigenvalue $+1$.
The expectation value $\langle\sigma_x\tau_x\rangle = \langle T_2|T_2\rangle = 1.$
$\langle\sigma_y\tau_y\rangle$: $\sigma_y\tau_y|T_2\rangle = -\frac{1}{\sqrt{2}}(|dd\rangle + |uu\rangle) = -|T_2\rangle.$
$|T_2\rangle$, is an eigenvector of the operator $\sigma_y\tau_y$ with eigenvalue -1.
The expectation value $\langle\sigma_y\tau_y\rangle = -\langle T_2|T_2\rangle = -1.$
Furthermore $\langle\sigma_x\rangle = 0$:
$\langle\sigma_x\rangle = \langle T_2|\sigma_x|T_2\rangle = \frac{1}{2}(\langle uu| + \langle dd|)(|du\rangle + |ud\rangle) = \frac{1}{2}(0 + 0 + 0 + 0) = 0.$
Similarly
$\langle\sigma_y\rangle = 0$, and $\langle\sigma_z\rangle = 0.$
Then, e.g.,
$\langle\sigma_x\tau_x\rangle - \langle\sigma_x\rangle\langle\tau_x\rangle = 1$
Consequently, we have *perfect* correlation when maximally correlated.
$|T_3\rangle$:
$\langle\sigma_z\tau_z\rangle$:
$\sigma_z\tau_z|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = |T_3\rangle.$
We find $|T_3\rangle$, to be an eigenvector of the operator $\sigma_z\tau_z$ with eigenvalue $+1$. The expectation value $\langle\sigma_z\tau_z\rangle = \langle T_3|T_3\rangle = 1.$
$\langle\sigma_x\tau_x\rangle$: $\sigma_x\tau_x|T_3\rangle = \frac{1}{\sqrt{2}}(|dd\rangle - |uu\rangle) = -|T_3\rangle;$
$|T_3\rangle$, is an eigenvector of the operator $\sigma_x\tau_x$ with eigenvalue -1.
The expectation value $\langle\sigma_x\tau_x\rangle = -\langle T_3|T_3\rangle = -1.$
$\langle\sigma_y\tau_y\rangle$: $\sigma_y\tau_y|T_3\rangle = \frac{1}{\sqrt{2}}(|uu\rangle - |dd\rangle) = |T_3\rangle.$
$|T_3\rangle$, is an eigenvector of the operator $\sigma_y\tau_y$ with eigenvalue $+1$.
The expectation value $\langle\sigma_y\tau_y\rangle = \langle T_3|T_3\rangle = 1.$
Furthermore $\langle\sigma_x\rangle = 0$:
$\langle\sigma_x\rangle = \langle T_3|\sigma_x|T_3\rangle = \frac{1}{2}(\langle uu| - \langle dd|)(|du\rangle - |ud\rangle) = \frac{1}{2}(0 - 0 + 0 + 0) = 0.$
Similarly
$\langle\sigma_y\rangle = 0$, and $\langle\sigma_z\rangle = 0.$
Then, e.g.,
$\langle\sigma_y\tau_y\rangle - \langle\sigma_y\rangle\langle\tau_y\rangle = 1$
Consequently, we have *perfect* correlation when maximally correlated.

Susskind paid attention to the measurement of the observable $\vec{\sigma}\cdot\vec{\tau}$. The operator is used in the next exercise. On this page Susskind paid attention to commutation: *"The problem is that Bob cannot simultaneously measure the individual components of τ, because they do not commute".* See the remark on page 176.

Exercise 6.9 The eigenvectors of the product operator and the entangled vectors

Prove that the four vectors $|sing\rangle$, $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$ are eigenvectors of $\vec{\sigma}\cdot\vec{\tau}$. What are their eigenvalues?
Most of the work has been done in Exercises 6.6, 6.7 and 6.8.
We must complete the calculus for $|sing\rangle$. On page 177 and page 178 Susskind calculated $\sigma_z\tau_z|sing\rangle = -\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) = -|sing\rangle$,
and $|sing\rangle$, is an eigenvector of the operator $\sigma_z\tau_z$ with eigenvalue -1.
$\sigma_x\tau_x|sing\rangle = \frac{1}{\sqrt{2}}(|du\rangle - |ud\rangle) = -|sing\rangle$.
So, $|sing\rangle$, is an eigenvector of the operator $\sigma_x\tau_x$ wit eigenvalue -1.
The expectation value $\langle\sigma_z\tau_z\rangle = -\langle sing|sing\rangle = -1$, and the expectation value $\langle\sigma_x\tau_x\rangle = -\langle sing|sing\rangle = -1$.
For $\sigma_y\tau_y|sing\rangle = \frac{1}{\sqrt{2}}(|du\rangle - |ud\rangle) = -|sing\rangle$,
and $\langle\sigma_y\tau_y\rangle = -\langle sing|sing\rangle = -1$.
$|sing\rangle$, is an eigenvector of the operator $\sigma_y\tau_y$ with eigenvalue -1.
1) Is $|sing\rangle$, an eigenvector of the operator $\vec{\sigma}\cdot\vec{\tau}$?
We know $\vec{\sigma}\cdot\vec{\tau} = \sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z$, (page 180).
When we operate $\vec{\sigma}\cdot\vec{\tau}$ on $|sing\rangle$, we have
$(\sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z)|sing\rangle = -3|sing\rangle$.
So,
$|sing\rangle$, is an eigenvector of $\vec{\sigma}.\vec{\tau}$ with eigenvalue -3.
2) With the results of Exercises 6.7 and 6.8, it follows straightforward $|T_1\rangle$, $|T_2\rangle$, and $|T_3\rangle$ to be eigenvectors of $\vec{\sigma}\cdot\vec{\tau}$, with eigenvalues the same and equal to $+1(= 1 + 1 - 1)$.
Note: the eigenvalues of the components of $\vec{\sigma}\cdot\vec{\tau}$ vary with the triplets. See Exercise 6.8.

Remark: On page 181 Susskind writes: *"……., the singlet is an eigenvector, and the triplets are all eigenvectors with a different degenerate eigenvalue"*. 'Different degenerate eigenvalue', what kind of animal is that? On page 64 Susskind writes: *"……. This situation, where two different eigenvectors have the same eigenvalue, has a name: it's called degeneracy"*. On page 67 Susskind writes: *"…. This typically happens when a system has degenerate states-distinct states that have the same eigenvalue"*. So, the expression 'different degenerate eigenvalue' is not clear to me. May be a typo?

Exercise 6.10 The energies and eigenvectors of two spins for a given Hamiltonian.

A system of two spins has the Hamiltonian $\boldsymbol{H} = \frac{\omega}{2}\vec{\sigma}\cdot\vec{\tau}$.

(Note : Hamiltonian without Planck's constant).

What are the possible energies of the system, and what are the eigenvectors of the Hamiltonian? Suppose the system starts in the state $|uu\rangle$. What is the state at any later time? Answer the same question for initial states of $|ud\rangle$, $|du\rangle$, and $|dd\rangle$.

I follow the steps of the recipe for a Schrödinger ket, page 124, (See also Exercise 4.6):

1) Derive,…., the Hamiltonian operator $\boldsymbol{H} \to \boldsymbol{H} = \frac{\omega}{2}\vec{\sigma}\cdot\vec{\tau}$.

2) Prepare an initial state $|\Psi(0)\rangle : |uu\rangle$.

3) Find the eigenvalues and eigenvectors of $\boldsymbol{H}$ Schrödinger equation, $\boldsymbol{H}\big|E_j\big\rangle = E_j\big|E_j\big\rangle$.

I can choose as eigenvectors the singlet and triplets. They form a complete set of orthonormal vectors in the four-dimensional vector space. The eigenvalues E_j are $-\frac{3\omega\hbar}{2}$ for $|sing\rangle$ and $\frac{\omega\hbar}{2}$ for the triplets.

I include here Planck's constant $\hbar$ since I consider the absence of $\hbar$ in the Hamiltonian of this Exercise a printing error.

Could we also have taken the vectors $|uu\rangle$, $|ud\rangle, |du\rangle,$ and $|dd\rangle$, as a basis? Do they also form a complete set of orthonormal vectors? They comprise the basis set for the four-dimensional vector space. In the note below I will pay attention to this question. I stay with the singlet and triplets being the eigenvectors of $\vec{\sigma}\cdot\vec{\tau}$.

4) Use the initial state-vector $|\Psi(0)\rangle$, along with the eigenvectors $|E_j\rangle$ from step 3), to calculate the initial coefficient $\alpha_j(0) = \big\langle E_j\big|\Psi(0)\big\rangle = \big\langle E_j\big|uu\big\rangle$.

$\alpha_1(0) = \langle E_1|uu\rangle = \langle sing|uu\rangle = 0$,

$\alpha_2(0) = \langle E_2|uu\rangle = \langle T_1|uu\rangle = 0$,

$\alpha_3(0) = \langle E_3|uu\rangle = \langle T_2|uu\rangle = \frac{1}{\sqrt{2}}$, and

$\alpha_4(0) = \langle E_4|uu\rangle = \langle T_3|uu\rangle = \frac{1}{\sqrt{2}}$.

To find these coefficient use has been made of the expressions for the singlet and triplets given in the above exercises and the Kronecker delta.

5) Rewrite $|\Psi(0)\rangle$ in terms of the four eigenvectors $|E_j\rangle$, and the initial coefficients $\alpha_j(0)$: $|\Psi(0)\rangle = \sum_j \alpha_j(0)|E_j\rangle$,

6) In the above equation, replace $\alpha_j(0)$ with $\alpha_j(t)$ to capture its time-dependence. As a result $|\Psi(t)\rangle$: $|\Psi(t)\rangle = \sum_j \alpha_j(t)|E_j\rangle$.

7) Using Eq. (4.30), replace $\alpha_j(t)$ with $\alpha_j(0)e^{-iE_jt/\hbar}$:

$|\Psi(t)\rangle = \sum_j \alpha_j(0)e^{-\frac{iE_jt}{\hbar}}|E_j\rangle$.

8) With the above results for $\alpha_j(0)$ and eigenvalues E_j , $-\frac{3\omega\hbar}{2}$ and $\frac{\omega\hbar}{2}$:

$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}(|T_2\rangle + |T_3\rangle) = e^{-\frac{i\omega t}{2}}(|uu\rangle)$.

In this expression the eigenvalue $-\frac{3\omega\hbar}{2}$ did not contribute. This eigenvalue will contribute for the initial states $|ud\rangle$ and $|du\rangle$.

Now, the same exercise with $|ud\rangle$ as initial state. The singlet and triplets are again used as eigenvectors and start with step 4) of the recipe.

4) Use the initial state-vector $|\Psi(0)\rangle$, along with the eigenvectors $|E_j\rangle$ from step 3), to calculate the initial coefficient $\alpha_j(0) = \big\langle E_j\big|\Psi(0)\big\rangle = \big\langle E_j\big|ud\big\rangle$.

$\alpha_1(0) = \langle E_1|ud\rangle = \langle sing|ud\rangle = \frac{1}{\sqrt{2}}$,

$\alpha_2(0) = \langle E_2|ud\rangle = \langle T_1|ud\rangle = \frac{1}{\sqrt{2}}$,

$\alpha_3(0) = \langle E_3|ud\rangle = \langle T_2|ud\rangle = 0$, and

$\alpha_4(0) = \langle E_4|ud\rangle = \langle T_3|ud\rangle = 0$.

To find these coefficient use has been made of the expressions for the singlet and triplets given in the above exercises and the Kronecker delta.

5) Rewrite $|\Psi(0)\rangle$ in terms of the four eigenvectors $|E_j\rangle$, and the initial coefficients $\alpha_j(0)$: $|\Psi(0)\rangle = \sum_j \alpha_j(0)|E_j\rangle$,

6) In the above equation, replace $\alpha_j(0)$ with $\alpha_j(t)$ to capture its time-dependence. As a result $|\Psi(t)\rangle$: $|\Psi(t)\rangle = \sum_j \alpha_j(t)|E_j\rangle$.

7) Using Eq. (4.30), replace $\alpha_j(t)$ with $\alpha_j(0)e^{-iE_jt/\hbar}$:

$|\Psi(t)\rangle = \sum_j \alpha_j(0) e^{-\frac{iE_j t}{\hbar}} |E_j\rangle$.

8) With the above results for $\alpha_j(0)$ and eigenvalues E_j , $-\frac{3\omega\hbar}{2}$ and $\frac{\omega\hbar}{2}$:

$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}[e^{\frac{3i\omega t}{2}}|sing\rangle + e^{-\frac{i\omega t}{2}}|T_1\rangle] = \frac{1}{2}\left[e^{\frac{3i\omega t}{2}}(|ud\rangle - |du\rangle) + e^{-\frac{i\omega t}{2}}(|ud\rangle + |du\rangle)\right]$.

$|\Psi(t)\rangle = \frac{1}{2}e^{-\frac{i\omega t}{2}}((e^{2i\omega t}+1)|ud\rangle + (1-e^{2i\omega t})|du\rangle)$.

If we calculate the probability to find the result for the singlet, we find : $\frac{1}{2}$, independent of time.

Now at time t we have a mixed state.

For the initial state $|du\rangle$ we again find a mixed state at time t:

$|\Psi(t)\rangle = \frac{1}{2}e^{-\frac{i\omega t}{2}}((e^{2i\omega t}+1)|du\rangle + (1-e^{2i\omega t})|ud\rangle)$.

For the initial state $|dd\rangle$ we find

$|\Psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-\frac{i\omega t}{2}}(|T_2\rangle - |T_3\rangle) = e^{-\frac{i\omega t}{2}}(|dd\rangle)$.

Note:

$\vec{\sigma}\cdot\vec{\tau}|uu\rangle = |uu\rangle$,

$\vec{\sigma}\cdot\vec{\tau}|dd\rangle = |dd\rangle$,

$\vec{\sigma}\cdot\vec{\tau}|ud\rangle = -|ud\rangle + 2|du\rangle$,

and

$\vec{\sigma}\cdot\vec{\tau}|du\rangle = -|du\rangle + 2|ud\rangle$.

$|uu\rangle, |dd\rangle, |ud\rangle$, and $|du\rangle$ do not constitute a complete set of eigenvectors since $|ud\rangle$ and $|du\rangle$ are no eigenvectors of $\vec{\sigma}\cdot\vec{\tau}$.

$\vec{\sigma}\cdot\vec{\tau}(|ud\rangle + |du\rangle) = |ud\rangle + |du\rangle = \sqrt{2}|T_1\rangle$,

and $|ud\rangle + |du\rangle$ is an eigenvector of $\vec{\sigma}\cdot\vec{\tau}$.

Likewise, $|ud\rangle - |du\rangle$ is an eigenvector of $\vec{\sigma}.\vec{\tau}$.

Then,

$|uu\rangle, |dd\rangle, (|ud\rangle + |du\rangle)$ and $(|ud\rangle - |du\rangle)$ constitute a complete set of eigenvectors of the observable $\vec{\sigma}\cdot\vec{\tau}$.

The observable $\vec{\sigma}\cdot\vec{\tau}$ is written *"as the ordinary dot product of the vector operators $\vec{\sigma}$ and $\vec{\tau}$: $\vec{\sigma}\cdot\vec{\tau} = \sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z$(page 180)".* You could be tempted to multiply the Pauli matrices. However, then a 2×2 matrix is found: $\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$. With the combined states we have a 4×4 vector space and by applying this 2×2 matrix you are lost. Lost in temptation, sort of.

$\sigma_x\tau_x = \sigma_x \otimes \tau_x = \begin{pmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{pmatrix}$, (use has been made of the tensor product $\otimes$ as explained in Lecture 7 and illustrated in the section of the elegance of tensor products in my notes above page 40).

$\sigma_y\tau_y = \sigma_y \otimes \tau_y = \begin{pmatrix}0&0&0&-1\\0&0&1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}$ and

$$\sigma_z\tau_z = \sigma_z \otimes \tau_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently $\vec{\sigma} \cdot \vec{\tau}$:

$$\vec{\sigma} \cdot \vec{\tau} = \sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Lecture 7. More on entanglement.

This lecture elaborates further on entanglement by using tensor product matrices. A new operator, the outer product is introduced along with the density matrix.

7.1 Mathematical Interlude: Tensor Products in Component Form

7.1.1 Building Tensor Product Matrices from Basic Principles

Lecture 3.1.1 Rehearsal.

Eqs. (3.1)-(3.4):

$\sum_j \langle k|\boldsymbol{M}|j\rangle\, \alpha_j = \sum_j \beta_j \langle k|j\rangle,$

$$\boldsymbol{M} = \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{23} & m_{33} \end{pmatrix},$$

$\sum_j \langle k|\boldsymbol{M}|j\rangle\, \alpha_j = \sum_j m_{kj}\, \alpha_j = \beta_k \rightarrow m_{kj} = \langle k|\boldsymbol{M}|j\rangle,$

where m_{kj} are called the matrix elements,

and

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{23} & m_{33} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}.$$

Susskind illustrated the tensor product matrix with the example $\sigma_z \otimes I$.

In Eq. (7.2), the components of the two-spin vector are u and d. Then, with the 4 , combined, basis vectors (see page 185) a 4×4 matrix is constructed. Use has been made of $\langle ab|a'b'\rangle = \delta_{aa'}\delta_{bb'}$, to obtain the matrix Eq.(7.4).

Page 187, with Eqs. (7.4) and (7.5):

$$\sigma_z \otimes I|du\rangle = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = -|du\rangle\ .$$

7.1.2 Building Tensor Product Matrices from Component Matrices

In Eqs. (7.6) and (7.7), Susskind presented the recipe for tensor products.

On the pages 189-191, Susskind presented various examples of tensor products.

To summarize: *"…..it is about matrix representation of abstract operators and state-vectors that replicates their known behavior."*

Exercise 7.1: Operate Bob's tensor on the basis vectors

Write the tensor product $I \otimes \tau_x$ as a matrix and apply that matrix to each of the $|uu\rangle$, $|ud\rangle$, $|du\rangle$, and $|dd\rangle$ column vectors. Show that Alice's half of the state vector is unchanged in each case. Recall that I is the 2×2 unit matrix.

$I \otimes \tau_x = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0\,1\,0\,0\\ 1\,0\,0\,0\\ 0\,0\,0\,1\\ 0\,0\,1\,0\end{pmatrix}$, and

$|uu\rangle = \begin{pmatrix}1\\ 0\end{pmatrix} \otimes \begin{pmatrix}1\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix}$.

Now $I \otimes \tau_x|uu\rangle = \begin{pmatrix}0\,1\,0\,0\\ 1\,0\,0\,0\\ 0\,0\,0\,1\\ 0\,0\,1\,0\end{pmatrix}\begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix} = |ud\rangle$, page 189 Eqs.(7.9).

Then $I \otimes \tau_x|ud\rangle = \begin{pmatrix}0\,1\,0\,0\\ 1\,0\,0\,0\\ 0\,0\,0\,1\\ 0\,0\,1\,0\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\\ 0\end{pmatrix} = \begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix} = |uu\rangle$.

Along the same lines we find $I \otimes \tau_x|du\rangle = |dd\rangle$ and $I \otimes \tau_x|dd\rangle = |du\rangle$. This shows that Alice's half of the state vector is unchanged in each case.

Do not be confused by this statement:

$I \otimes \sigma_x = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \otimes \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0\,1\,0\,0\\ 1\,0\,0\,0\\ 0\,0\,0\,1\\ 0\,0\,1\,0\end{pmatrix}$, giving the same results as obtained with $I \otimes \tau_x$.

Exercise 7.2 Matrix element of the tensor product of both spin operators

Calculate the matrix elements of $\sigma_z \otimes \tau_x$ by forming inner products as we did in Eq. 7.2.

$$\begin{pmatrix}\langle uu|\sigma_z\tau_x|uu\rangle & \langle uu|\sigma_z\tau_x|ud\rangle & \langle uu|\sigma_z\tau_x|du\rangle & \langle uu|\sigma_z\tau_x|dd\rangle\\ \langle ud|\sigma_z\tau_x|uu\rangle & \langle ud|\sigma_z\tau_x|ud\rangle & \langle ud|\sigma_z\tau_x|du\rangle & \langle ud|\sigma_z\tau_x|dd\rangle\\ \langle du|\sigma_z\tau_x|uu\rangle & \langle du|\sigma_z\tau_x|ud\rangle & \langle du|\sigma_z\tau_x|du\rangle & \langle du|\sigma_z\tau_x|dd\rangle\\ \langle dd|\sigma_z\tau_x|uu\rangle & \langle dd|\sigma_z\tau_x|ud\rangle & \langle dd|\sigma_z\tau_x|du\rangle & \langle dd|\sigma_z\tau_x|dd\rangle\end{pmatrix}.$$

Up to learning about the tensor product, we assume σ_z to operate on the leftmost (that is Alice's) state-label (Susskind) and τ_x on the rightmost (that is Bob's) state-label. That assumption is still correct. However, now we know how to construct the matrix $\sigma_z\tau_x$ and the combined states $|uu\rangle$, etc, Eqs.(7.2) and (7.5), resulting from the tensor product. Furthermore, using the fact that these states are orthonormal or, and that is the same, make use of the Kronecker delta for combined states, $\langle ab|a'b'\rangle = \delta_{aa'}\delta_{bb'}$, the above matrix becomes:

$$\begin{pmatrix}0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & -1 & 0\end{pmatrix} = \sigma_z \otimes \tau_x.$$

Exercise 7.3 More on tensor products

a) Rewrite Eq. (7.10), $(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$, in component form, replacing the symbols $A, B, a,$ and b with the matrices and column vectors of Eqs. (7.7) and (7.8).

A bit tedious indeed!

We have $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, $a = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}$ and $b = \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix}$.

$A \otimes B$ equals Eq. (7.7) and $a \otimes b$ equals Eq. (7.8).

$$A \otimes B = \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix},$$

and

$$a \otimes b = \begin{pmatrix} a_{11}b_{11} \\ a_{11}b_{21} \\ a_{21}b_{11} \\ a_{21}b_{21} \end{pmatrix}.$$

The left-hand side of Eq. (7.10) is a matrix multiplied into a column vector producing a new column vector or a 4×1 matrix.

b) Perform the matrix multiplications Aa and Bb on the right-hand side of Eq. (7.10) .

Aa is a 2×1 matrix: $Aa = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} = \begin{pmatrix} A_{11}a_{11} + A_{12}a_{21} \\ A_{21}a_{11} + A_{22}a_{21} \end{pmatrix}$.

Bb is a 2×1 matrix: $Bb = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} b_{11} \\ b_{21} \end{pmatrix} = \begin{pmatrix} B_{11}b_{11} + B_{12}b_{21} \\ B_{21}b_{11} + B_{22}b_{21} \end{pmatrix}$.

Tensor multiplications of these 2×1 matrices produces a 4×1 matrix.

$$Aa \otimes Bb = \begin{pmatrix} [A_{11}a_{11} + A_{12}a_{21}] \cdot [B_{11}b_{11} + B_{12}b_{21}] \\ [A_{11}a_{11} + A_{12}a_{21}] \cdot [B_{21}b_{11} + B_{22}b_{21}] \\ [A_{21}a_{11} + A_{22}a_{21}] \cdot [B_{11}b_{11} + B_{12}b_{21}] \\ [A_{21}a_{11} + A_{22}a_{21}] \cdot [B_{21}b_{11} + B_{22}b_{21}] \end{pmatrix}$$

c) Expand all three *Kronecker products*, Eqs.(7.6) and (7.7), the matrix version of the tensor products in Eq.(7.10).

We have all the ingredients for expanding

$(A \otimes B)(a \otimes b) = (Aa \otimes Bb)$

On the left-hand side we find a 4×1 matrix as we do on the right-hand side:

$$(A \otimes B)(a \otimes b) = \begin{pmatrix} A_{11}B_{11}a_{11}b_{11} + A_{11}B_{12}a_{11}b_{21} + A_{12}B_{11}a_{21}b_{11} + A_{12}B_{12}a_{21}b_{21} \\ A_{11}B_{21}a_{11}b_{11} + A_{11}B_{22}a_{11}b_{21} + A_{12}B_{21}a_{21}b_{11} + A_{12}B_{22}a_{21}b_{21} \\ A_{21}B_{11}a_{11}b_{11} + A_{21}B_{12}a_{11}b_{21} + A_{22}B_{11}a_{21}b_{11} + A_{22}B_{12}a_{21}b_{21} \\ A_{21}B_{21}a_{11}b_{11} + A_{21}B_{22}a_{11}b_{21} + A_{22}B_{21}a_{21}b_{11} + A_{22}B_{22}a_{21}b_{21} \end{pmatrix},$$

and

$$Aa \otimes Bb = \begin{pmatrix} A_{11}B_{11}a_{11}b_{11} + A_{11}B_{12}a_{11}b_{21} + A_{12}B_{11}a_{21}b_{11} + A_{12}B_{12}a_{21}b_{21} \\ A_{11}B_{21}a_{11}b_{11} + A_{11}B_{22}a_{11}b_{21} + A_{12}B_{21}a_{21}b_{11} + A_{12}B_{22}a_{21}b_{21} \\ A_{21}B_{11}a_{11}b_{11} + A_{21}B_{12}a_{11}b_{21} + A_{22}B_{11}a_{21}b_{11} + A_{22}B_{12}a_{21}b_{21} \\ A_{21}B_{21}a_{11}b_{11} + A_{21}B_{22}a_{11}b_{21} + A_{22}B_{21}a_{21}b_{11} + A_{22}B_{22}a_{21}b_{21} \end{pmatrix}.$$

d)Verify the row and column sizes of each Kronecker delta product of Eq. (7.10):

- $A \otimes B$: a tensor product of two 2×2 matrices produces a 4×4 matrix represented by Eq. (7.7).
- $a \otimes b$: a tensor product of two 2×1 matrices gives a 4×1 matrix represented by Eq. (7.8).
- $Aa \otimes Bb$: a tensor product of two 2×1 matrices as explained above in b). This product results in a 4×1 matrix.

  Remark: "$Aa \otimes Bb$: 4×4". A printing error, see e) and f).

e) Perform the matrix multiplication on the left-hand side in Eq. (7.10), resulting in a 4×1 column vector. Each row should be the sum of four separate terms. This is shown under c).

f) Finally, verify that the resulting column vector on the left hand and right sides are identical. This is shown under c).

Example tensor product Interlude:

Now, before to continue with Lecture 7, let us return to **Exercise 6.10** and apply the above toolkit of tensor algebra to find the eigenvalues of the operator $\vec{\sigma}\cdot\vec{\tau} = \sigma_x\tau_x + \sigma_y\tau_y + \sigma_z\tau_z$. Then we must realize that we construct the tensor representation of the right-hand side of the expression for $\vec{\sigma}\cdot\vec{\tau}$. Using the 2×2 Pauli matrices we have:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

a 4×4 matrix of which the eigenvalues λ_i are found from the determinant leading to the polynomial: $(1-\lambda)^2((1+\lambda)^2-4)=0$, with three roots:

$\lambda_i = 1$(degeneracy) and one root $\lambda_i = -3$.

In accordance with the eigenvalues of the singlet and triplets. The difference between the eigenvalues of the singlet and triplets and of the other orthonormal basis $|uu\rangle, |ud\rangle, |du\rangle$, and$|dd\rangle$, is:

the latter basis being the orthonormal basis for the four-dimensional space, does not form a complete set of eigenvectors of the operator $\vec{\sigma}\cdot\vec{\tau}$ as mentioned before. In the Exercise 6.10 we presumed the singlet and the triplet to be the eigenvectors of $\vec{\sigma}\cdot\vec{\tau}$ and it worked. Now, given the above matrix and its eigenvalues, can we find the eigenvectors in a general way? Let's start with the eigenvalue -3 and assume the components of the column vector representation of the eigenvector to be a, b, c and d. Then:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = -3\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

applying matrix vector multiplication and equating:

$a = -3a, \rightarrow a = 0, d = -3d, \rightarrow d = 0$, and $b = -c$.

With normalization, $bb^* + cc^* = 1, b = \frac{1}{\sqrt{2}}$, and $c = -\frac{1}{\sqrt{2}}$.

Now, the eigenvector is

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

Look on page 189, Eq. (7.9) and we find that this eigenvector is represented by

$\frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle) = |sing\rangle$.

What about the three eigenvectors with eigenvalue $+1$?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix},$$

applying matrix vector multiplication and equating:$a = a, d = d$, and $b = c$.

The eigenvectors are $\begin{pmatrix} a \\ b \\ b \\ d \end{pmatrix}$ and the normalization condition is:

$a^*a + 2b^*b + d^*d = 1$.

Furthermore, we have the column vector representations of the basis vectors given on pages 189 and 190, Eq.(7.9).
Starting with a simple eigenvector where $a = 0$ and $d = 0$, and b is real, we find for the first eigenvector with eigenvalue $+1$ the triplet:
$|T_1\rangle = \frac{1}{\sqrt{2}}(|ud\rangle + |du\rangle)$.
The other two are found by $b = 0$ and $a = d$, and $b = 0$ and $a = -d$. This produces the eigenvectors $|T_2\rangle$ and $|T_3\rangle$.
By the way, set $a = 1$, and $b = c = d = 0$ and you have $|uu\rangle$. Furthermore, set $d = 1$, and $a = b = d = 0$ giving $|dd\rangle$. In Fitzpatrick (Undergraduate course) $|uu\rangle$ and $|dd\rangle$ with $|T_1\rangle$ are the three triplets. This is no contradiction, since adding and subtracting $|uu\rangle$ and $|dd\rangle$ respectively, with normalization, give us $|T_2\rangle$ and$|T_3\rangle$.
In Noordzij, some additional thoughts are presented on the construction of the triplets.
End of tensor product example.

7.2 Mathematical Interlude: Outer Product

In this Lecture Susskind introduced the outer product $|\psi\rangle\langle\phi|$ as a new linear operator.
On page 34 we encountered something like an outer product: $|i\rangle\langle i|$.

Dirac(page 25) proved the outer product to be a linear operator.

At the top of page 194: *"….we take the inner product of $\langle\phi|$ with $|A\rangle$ (the result is a complex number) and multiply it by the ket $|\psi\rangle$."* → And a new ket is obtained. So, the outer product is an operator.
On page 194 Susskind gives some properties of projection operators-a special form of outer product- that "*can easily be proved*"(Susskind uses $|\psi\rangle$, I prefer $|\Psi\rangle$):
Projection operators are Hermitian.

***Remark**: Dirac proved on page 28 the complex conjugate of the outer product to be equal to the outer product. Dirac: "Multiplying $|A\rangle\langle B|$ into a general bra $\langle P|$ we get $\langle P|A\rangle\langle B|$, whose conjugate imaginary ket is $(\langle P|A\rangle\langle B|)^* = \langle P|A\rangle^*|B\rangle = |B\rangle\langle A|P\rangle$, and $(\langle P|A\rangle\langle B|)^* = |P\rangle(|A\rangle\langle B|)^* = |B\rangle\langle A|P\rangle$. This gives for any $|P\rangle$: $(|A\rangle\langle B|)^* = |B\rangle\langle A|$.
Well , to be Hermitian, the projection operator must be a matrix. With a column vector representation of the ket in the projection operator we find a matrix. Then, with the proof of Dirac equating $A = B = \psi$ and the matrix representation of the outer product it appears:*$(|\psi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\psi|$.

Properties of projection operators:

- Projection operators are Hermitian (See Dirac, page 28).
  Remark: On page 28, Eq. (7) Dirac proved the conjugate imaginary of $|A\rangle\langle B|$, $\overline{|A\rangle\langle B|}$, to be equal to $|B\rangle\langle A|$. See my ***Remark*** above: $(|\psi\rangle\langle\psi|)^\dagger = |\psi\rangle\langle\psi|$.
- Any vector $|\phi\rangle$ orthogonal to $|\psi\rangle$ is an eigenvector of $|\psi\rangle\langle\psi|$ with eigenvalue zero:
  $|\psi\rangle\langle\psi|\ |\phi\rangle = |\psi\rangle\ \langle\phi|\psi\rangle = 0|\psi\rangle$.
- The vector $|\psi\rangle$ is an eigenvector of its projection operator with eigenvalue 1:
  $|\psi\rangle\langle\psi|\ \ \psi\rangle = |\psi\rangle\langle\psi|\psi\rangle = |\psi\rangle$, since $|\psi\rangle$ is normalized; $\langle\psi|\psi\rangle = 1$.

- The square of a projection operator is the same as the projection operator itself: $(|\psi\rangle\langle\psi|)^2 = |\psi\rangle\langle\psi|\,|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi|$.
- The trace of an operator $\boldsymbol{L}$ is, in matrix representation, $Tr\,\boldsymbol{L} = \sum_i \langle i|\boldsymbol{L}|i\rangle$, which is just the sum of $\boldsymbol{L}$'s diagonal matrix elements.
  The trace of a projection operator is 1:
  $Tr|\psi\rangle\langle\psi| = \sum_i \langle i|\psi\rangle\langle\psi|i\rangle$, the sum of the diagonal matrix elements.
  $Tr|\psi\rangle\langle\psi| = \sum_i \langle i|\psi\rangle\langle\psi|i\rangle = \sum_i \langle\psi|i\rangle\langle i|\psi\rangle = \langle\psi|\psi\rangle = 1$, since $|\psi\rangle$ is normalized. Or, since the projection operator is Hermitian and can be diagonalized and there is only one eigenvector with unit eigenvalue(third bullet point above), the trace is the sum of its eigenvalues.
- Add all the projection operators for a basis system and, we find $\sum_i |i\rangle\langle i| = \boldsymbol{I}$.
  Remember Lecture 1.95, page 34.
  Consider the vector $|A\rangle$ written in the basis form
  $|A\rangle = \sum_i \alpha_i |i\rangle$,
  rewritten in the elegant form, $\alpha_i = \langle i|A\rangle$,
  $|A\rangle = \sum_i |i\rangle\langle i|A\rangle$.
  Then,
  $|A\rangle = (\sum_i |i\rangle\langle i|)|A\rangle$.
  So, $\sum_i |i\rangle\langle i| = \boldsymbol{I}$.
  Or : $\sum_i |i\rangle\langle i| \sum_k \alpha_k |k\rangle = \sum_i |i\rangle \sum_k \alpha_k \langle i|k\rangle = \sum_i |i\rangle\alpha_i = |A\rangle$, and consequently $\sum_i |i\rangle\langle i| = \boldsymbol{I}$.
  Remember: $\sum_k \alpha_k \langle i|k\rangle = \alpha_i$, where use has been made of $\delta_{ik} = \langle i|k\rangle$.
  See also my notes on section 3.1.5.
  The proof can be found in Dirac page 63.
  At the end of this section, Susskind presented Eq.(7.12), and proved a theorem about projection operators, *a special case of the outer product*, and expectation values,
  Eq.(7.12): $\langle\psi|\boldsymbol{L}|\psi\rangle = Tr|\psi\rangle\langle\psi|\boldsymbol{L}$.
  There is more:

7.2.1 The projection operator and the Gram-Schmidt Procedure

In Lecture 3.1.6 the Gram-Schmidt Procedure is presented and explained. Here, the projection operator (outer product) comes into play.
"*Sometimes we encounter a set of linearly independent eigen vectors that do not form an orthonormal set.*", page 67.
So, we have two kets, normalized:
$|\alpha_1\rangle$ and $|\alpha_2\rangle$, with $\langle\alpha_1|\alpha_2\rangle \neq 0$.
With the projection operator, $|\alpha_1\rangle\langle\alpha_1|$, construct a new ket:
$|\alpha_3\rangle = |\alpha_2\rangle - |\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle$.
Then,
$\langle\alpha_1|\alpha_3\rangle = \langle\alpha_1|\alpha_2\rangle - \langle\alpha_1|\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle = \langle\alpha_1|\alpha_2\rangle - \langle\alpha_1|\alpha_2\rangle = 0$,
where use has been made of
$\langle\alpha_1|\alpha_1\rangle = 1$.
The new ket $|\alpha_3\rangle$ presents an orthogonal set with $|\alpha_1\rangle$. However, a orthonormal basis set is

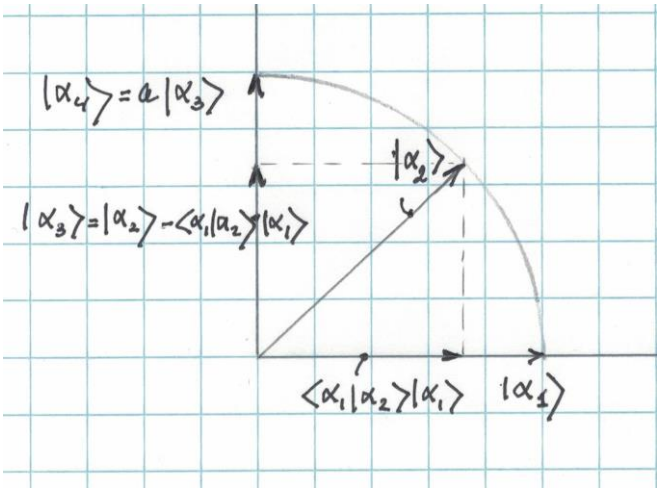
what we are looking for.
Let us introduce a number $a \Longrightarrow |a|^2\langle\alpha_3|\alpha_3\rangle = 1$.
The new ket $|\alpha_4\rangle = a|\alpha_3\rangle$, with $|\alpha_1\rangle$, is the new basis set $\{|\alpha_1\rangle, |\alpha_4\rangle\}$.
What about a? With $|\alpha_3\rangle = |\alpha_2\rangle - |\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle$:

$$a\,|\alpha_3\rangle = a|\alpha_2\rangle - a|\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle \Longrightarrow$$
$$\Longrightarrow [a^*\langle\alpha_2| - a^*\langle\alpha_1|\langle\alpha_1|\alpha_2\rangle^*][\,a|\alpha_2\rangle - a|\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle] =$$
$$|a|^2[\langle\alpha_2|\alpha_2\rangle - \langle\alpha_1|\alpha_2\rangle\langle\alpha_1|\alpha_2\rangle^* - \langle\alpha_2|\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle + \langle\alpha_1|\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle\langle\alpha_1|\alpha_2\rangle^*] =$$
$$= |a|^2[1 - |\langle\alpha_1|\alpha_2\rangle|^2] = 1 \Longrightarrow$$
$$\Longrightarrow a = \frac{e^{-i\phi}}{[1-|\langle\alpha_1|\alpha_2\rangle|^2]^{1/2}},$$

where ϕ, represents phase ambiguity and $\langle\alpha_2|\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle = \langle\alpha_1|\alpha_2\rangle^*\langle\alpha_1|\alpha_2\rangle$.
Neglecting this ambiguity,

$$|\alpha_4\rangle = a|\alpha_3\rangle = \frac{|\alpha_2\rangle - |\alpha_1\rangle\langle\alpha_1|\alpha_2\rangle}{[1--|\langle\alpha_1|\alpha_2\rangle|^2]^{1/2}}.$$

The new basis set, $\{|\alpha_1\rangle, |\alpha_4\rangle\}$, has been created.
As an illustration, I visualized the new basis set in the following Figure:



Gram-Schmidt Procedure and projection operator (See Figure 3.1 page 68, Susskind)

7.2.2 The Projection Operator and the Space of States

In the foregoing section we developed a set of basis vectors, $\{|\alpha_1\rangle, |\alpha_4\rangle\}$, for two-dimensional vector space. This is a complete set.
Now, let us operate a matrix A on this two-dimensional vector space. A has the aforementioned basis vectors as eigenvectors. The eigenvalues are denoted α_1, and α_4.
What are the elements of matrix A?
We have:

$$A|\alpha_1\rangle = \alpha_1|\alpha_1\rangle,$$

and

$$A|\alpha_4\rangle = \alpha_4|\alpha_4\rangle.$$

I use the column representation of the kets:

$$|\alpha_1\rangle = \begin{pmatrix} a \\ b \end{pmatrix}, \text{ and } |\alpha_4\rangle = \begin{pmatrix} c \\ d \end{pmatrix}.$$

The normalized kets give:

$$a^2 + b^2 = 1, \text{ and } c^2 + b^2 = 1.$$

The two vectors are orthonormal:
$ac + bd = 0$.
There is more:
multiply $A|\alpha_1\rangle = \alpha_1|\alpha_1\rangle$, to the right with the bra $\langle\alpha_1|$:
$A|\alpha_1\rangle\langle\alpha_1| = \alpha_1|\alpha_1\rangle\langle\alpha_1|$,
multiply $A|\alpha_4\rangle = \alpha_4|\alpha_4\rangle$, to the right with the bra $\langle\alpha_4|$:
$A|\alpha_4\rangle\langle\alpha_4| = \alpha_4|\alpha_4\rangle\langle\alpha_4|$.
Both expressions with projection operators are added:
$A(|\alpha_1\rangle\langle\alpha_1| + |\alpha_4\rangle\langle\alpha_4|) = \alpha_1|\alpha_1\rangle\langle\alpha_1| + \alpha_4|\alpha_4\rangle\langle\alpha_4|$.
As mentioned before $\{|\alpha_1\rangle, |\alpha_4\rangle\}$ is a complete set.
Consequently,
$(|\alpha_1\rangle\langle\alpha_1| + |\alpha_4\rangle\langle\alpha_4|) = I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.
Hence,
$A = \alpha_1|\alpha_1\rangle\langle\alpha_1| + \alpha_4|\alpha_4\rangle\langle\alpha_4|$.
With the column representation used in the projection operators, we find
$\begin{pmatrix}a^2 & ab\\ ba & b^2\end{pmatrix} + \begin{pmatrix}c^2 & cd\\ dc & d^2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$.
This gives another set of relations between the elements of the basis vectors:
$a^2 + c^2 = 1$, and $d^2 + b^2 = 1$,
$ab + cd = 0 \Leftrightarrow ba + dc = 0$.
There is some redundancy.
I found:
$|\alpha_1\rangle = \begin{pmatrix}a\\ b\end{pmatrix}$, and $|\alpha_4\rangle = \begin{pmatrix}b\\ -a\end{pmatrix}$.
With $b = \sqrt{1 - a^2}$, we are free to choose $a = 1$.
So,
$|\alpha_1\rangle = \begin{pmatrix}1\\ 0\end{pmatrix}$, and $|\alpha_4\rangle = \begin{pmatrix}0\\ -1\end{pmatrix}$. This choice could have been made from the start.
What about the matrix A?
$A = \alpha_1|\alpha_1\rangle\langle\alpha_1| + \alpha_4|\alpha_4\rangle\langle\alpha_4| = \alpha_1\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + \alpha_4\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$.
The elements of the matrix are the eigenvalues of the matrix:
$A = \begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_4\end{pmatrix}$.
Let's formulate this in a more general way.
We have the operator $\boldsymbol{L}$, and a complete set of kets $|\lambda_i\rangle$, spanning the vector space.
$\boldsymbol{L}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$,
where λ_i, are the eigenvalues and $|\lambda_i\rangle$, the eigenvectors of the operator $\boldsymbol{L}$.
Multiply $\boldsymbol{L}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle$, to the right on both sides with the bra $\langle\lambda_i|$,
$\boldsymbol{L}|\lambda_i\rangle\langle\lambda_i| = \lambda_i|\lambda_i\rangle\langle\lambda_i|$.
Take the sum over i, the complete set,
$\boldsymbol{L}\sum_i|\lambda_i\rangle\langle\lambda_i| = \sum_i \lambda_i\,|\lambda_i\rangle\langle\lambda_i| \Longrightarrow \boldsymbol{L} = \sum_i \lambda_i\,|\lambda_i\rangle\langle\lambda_i|$.

7.2.3 Expectation Value and Projection Operators

Assume a general ket $|\Psi\rangle$, expressed in the basis vectors of the complete set $|\lambda_i\rangle$:
$|\Psi\rangle = \sum_i c_i\,|\lambda_i\rangle$.
Now, we evaluate $\langle\Psi|\boldsymbol{L}|\Psi\rangle$, the expectation value measuring the ket $|\Psi\rangle$.
Above, we expressed the observable in projection operators:
$\boldsymbol{L} = \sum_i \lambda_i\,|\lambda_i\rangle\langle\lambda_i|$.
Then,
$\langle\Psi|\boldsymbol{L}|\Psi\rangle = \sum_i \lambda_i\langle\Psi\,|\lambda_i\rangle\langle\lambda_i|\Psi\rangle \Longrightarrow \langle\Psi|L|\Psi\rangle = \sum_{i,j}\lambda_i\,c_j^*\langle\lambda_j|\lambda_i\rangle c_j\langle\lambda_i|\lambda_j\rangle =$
$= \sum_{i,j}\lambda_i\,|c_j|^2|\langle\lambda_j|\lambda_i\rangle|^2$.
Now we see how the projection operators are "projecting out":
$\langle\Psi|\boldsymbol{L}|\Psi\rangle = \sum_{i,j}\lambda_i\,|c_j|^2\delta_{ij} = \sum_i \lambda_i|c_i|^2$.
Well, this a familiar expression:
$\langle\Psi|\boldsymbol{L}|\Psi\rangle = \sum_i \lambda_i|c_i|^2 = \sum_i \lambda_i\,P(\lambda_i)$ ⟹Eqs. (3.26) and (4.13).
With Eq.(7.12):
$\langle\Psi|\boldsymbol{L}|\Psi\rangle = \sum_i \lambda_i|c_i|^2 = \sum_i \lambda_i\,P(\lambda_i) = Tr|\Psi\rangle\langle\Psi|\boldsymbol{L}$.

7.3 Density Matrices: A New Tool

In this section, Susskind dealt with the case not having a complete knowledge of the state of a system.
The expectation value $\langle L\rangle$ in Eq. (7.13) should be written as $\langle\boldsymbol{L}\rangle(= Tr\;\rho\boldsymbol{L})$, a bold capital.
At the top of page 199, the matrix representation of the density matrix is given with respect to the basis $|a\rangle$:
$\rho_{aa'} = \langle a|\rho|a'\rangle$.
I suppose this to be a matrix element. What does the matrix look like? For a pure state?
ρ the density operator is the projection operator the basis of which is $|a\rangle$:
$\rho = |a\rangle\langle a|$.
$\rho|a\rangle = |a\rangle\langle a|a\rangle = |a\rangle$.
So, the eigenvalue of this operator is 1.
Consequently $Tr(\rho) = 1$,
and
$\langle a|\rho|a\rangle = \langle a|a\rangle\langle a|a\rangle = 1$.
Furthermore
$\langle\boldsymbol{L}\rangle = Tr\;\rho\boldsymbol{L} = \langle a|\boldsymbol{L}|a\rangle$**.**
In Lecture 7.6 an example is presented for an entangled state (a combined system).
The same question on matrix elements arises for the operator $\boldsymbol{L}$ and its matrix elements in Eq.(7.14).
Furthermore, on page 199, where the expectation value is presented in matrix representation, Eq.(7.14), the order of multiplication is changed compared to Eq.(7.13)(see also page 209). It does not matter. In addition, the order of the indices a and a' is important: in this way $Tr\;\rho\boldsymbol{L}$ is represented.
Let us pay some attention to the expression presented in Eq. (7.14):
$\langle\boldsymbol{L}\rangle = Tr\rho\boldsymbol{L} = \sum_{a,a'}\,L_{a',a}\,\rho_{a,a'}$**.**
The preceding expression is not easy "to read".

I choose for both operators 2×2 matrices. Both being Hamiltonian matrices:

$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$,

and

$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$.

Then,

$\langle \boldsymbol{L} \rangle = Tr\rho\boldsymbol{L} = Tr \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \cdot \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} = Tr \begin{pmatrix} \rho_{11}L_{11} + \rho_{12}L_{21} & \rho_{11}L_{12} + \rho_{12}L_{22} \\ \rho_{21}L_{11} + \rho_{22}L_{21} & \rho_{21}L_{12} + \rho_{22}L_{22} \end{pmatrix} =$

$= \rho_{11}L_{11} + \rho_{12}L_{21} + \rho_{21}L_{12} + \rho_{22}L_{22}$,

with,

$L_{21}^* = L_{12}$, and $\rho_{21}^* = \rho_{12}$.

$\langle \boldsymbol{L} \rangle = \rho_{11}L_{11} + \rho_{12}L_{12}^* + \rho_{12}^*L_{12} + \rho_{22}L_{22}$,

a real number.

Back to

$\langle \boldsymbol{L} \rangle = Tr\rho\boldsymbol{L} = \sum_{a,a'} L_{a',a}\, \rho_{a,a'}$.

With $a = 1,2$ and $a' = 1,2$

$\sum_{a,a'} L_{a',a}\, \rho_{a,a'} = L_{11}\rho_{11} + L_{12}\rho_{21} + L_{21}\rho_{12} + L_{22}\rho_{22}$.

Hence, this clarifies the notation.

Keep in mind: $TrAB = TrBA$,

for the square matrices A and B.

Definitions:

- *pure state*. When the density matrix corresponds to a single state, it is a projection operator that projects onto that state. Is a single state defined?
- *mixed state*. When the density matrix is a mixture of projection operators.
- *full composite system?*

7.4 Entanglement and Density Matrices

First the differences between classical mixed states and quantum mechanical entangled states are discussed.

Quantum Mechanics: *The state of a composite system can be absolutely pure, but each of its constituents must be described by a mixed state.*

Page 200, Susskind starts with a system composed of two parts A and B.

The wave function is given by Ψ. I will follow the convention of page 136 and use for the wave function ψ. So, I prefer Eq. (7.15) to be written like:

$\langle \boldsymbol{L} \rangle = \langle \Psi | \boldsymbol{L} | \Psi \rangle = \sum_{ab,a'b'} \psi^*(a'b') L_{a'b',ab} \psi(ab)$,

as did Susskind on page 206. On the right-hand side of Eq.(7.15), for the general expectation value, the expectation value is given in component representation.

Susskind paid attention to a situation where the observable $\boldsymbol{L}$ belongs to A. Then $\boldsymbol{L}$ acts trivially on the b-index. This leads to Eq. (7.16).

I think Susskind explained this more elegantly on page 204 and Eq. 7.18.

In Eq.(7.17), Susskind presented the matrix element of the density matrix, I suppose. How does this compare with the matrix element:

$\rho_{aa'} = \langle a | \rho | a' \rangle$,

at the top of page 199?

Remark:
Is $\rho_{aa'}$ a special density matrix where the influence of b is '*traced out*' or projected out.
In the literature the expression reduced density matrix can be found. Is $\rho_{aa'}$ a reduced density matrix?

Below Eq. (7.17) Susskind writes that "*Eq. (7.16) has exactly the same form as Eq. (7.14) for expectation value of a mixed state*". I assume that this applies for the situation where the elements of the density matrix $\rho_{aa'}$ are represented by Eq. (7.17).

Question: To me it is not clear that Eq.(7.16) has the same form as Eq. (7.14) for the expectation value of a mixed state.
Eq. (7.15):
$\langle \boldsymbol{L} \rangle = \langle \Psi | \boldsymbol{L} | \Psi \rangle = \sum_{ab,a'b'} \psi^*(a'b') L_{a'b',ab} \psi(ab)$.
A is not interested in B. Now the observable $\boldsymbol{L}$ is associated with A and the observable acts trivially on the b-index and $\langle \boldsymbol{L} \rangle$ is represented by Eq.(7.16):
$\langle \boldsymbol{L} \rangle = \sum_{a,b,a'} \psi^*(a'\, b) L_{a',a} \psi(ab)$.
We know $\rho_{aa'}$ to be, (Eq. 7.17):
$\rho_{aa'} = \sum_b \psi^*(a'\, b) \psi(ab)$.
With this expression for $\rho_{aa'}$, it is not clear to me Eq.(7.16) to have the same form as Eq.(7.14):
$\langle \boldsymbol{L} \rangle = \sum_{a,a'} L_{a',a}\, \rho_{a,a'}$.
Unless $\psi(ab)$ is a (complex) number and $L_{a',a}$ an element of a matrix, then Eq.(7.16) has the same form as Eq.(7.14). Otherwise?

Then, Susskind continues: *"Indeed, only in the very special case of a product state will ρ have the form of a projection operator".* Question: ρ of the product state or ρ of the subsystem?
Then, "*In other words, despite the fact that the composite system is described by a perfectly pure state, the subsystem A must be described by a mixed state".*
Pure and *mixed state* as defined in Lecture 7.3?

First, let's apply Eq.(7.17) for a product state given by Eq.(6.5):
$|\Psi\rangle = \alpha_u \beta_u |uu\rangle + \alpha_u \beta_d |ud\rangle + \alpha_d \beta_u |du\rangle + \alpha_d \beta_d |dd\rangle$,
and let's calculate ρ_{ud}.
Giving, with $a = u, a' = d$ and summing over all $b(= u, d)$:
$\rho_{ud} = \psi^*_{du}\psi_{uu} + \psi^*_{dd}\psi_{ud} \equiv \alpha^*_d \beta^*_u \alpha_u \beta_u + \alpha^*_d \beta^*_d \alpha_u \beta_d = \alpha^*_d \alpha_u (\beta^*_u \beta_u + \beta^*_d \beta_d)$.
With Eq.(6.4), normalization for a product state, we have
$\rho_{ud} = \alpha^*_d \alpha_u$,
Secondly, let us denote the operator for Alice ρ_A and for Bob ρ_B, and use the column representation of$|\Psi\rangle$, we obtain $\rho = \rho_A \otimes \rho_B$.
Note: Keep in mind the product state here to be a tensor product:
$|\Psi\rangle = \{\alpha_u |u\rangle + \alpha_d |d\rangle\} \otimes \{\beta_u |u\rangle + \beta_d |d\rangle\} = |\Psi_A\rangle \otimes |\Psi_B\rangle$
For completeness:

$$\rho_A = |\Psi_A\rangle \otimes \langle \Psi_A| = \begin{pmatrix} \alpha_u \\ \alpha_d \end{pmatrix} \otimes (\alpha^*_u \; \alpha^*_d) = \begin{pmatrix} \alpha^*_u \alpha_u & \alpha^*_d \alpha_u \\ \alpha^*_u \alpha_d & \alpha^*_d \alpha_d \end{pmatrix},$$

and

$$\rho_B = |\Psi_B\rangle \otimes \langle \Psi_B| = \begin{pmatrix} \beta_u \\ \beta_d \end{pmatrix} \otimes (\beta^*_u \; \beta^*_d) = \begin{pmatrix} \beta^*_u \beta_u & \beta^*_d \beta_u \\ \beta^*_u \beta_d & \beta^*_d \beta_d \end{pmatrix};$$

$$\rho = |\Psi\rangle \otimes \langle\Psi| = \begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} \otimes (\alpha_u^*\beta_u^* \;\; \alpha_u^*\beta_d^* \;\; \alpha_d^*\beta_u^* \;\; \alpha_d^*\beta_d^*).$$

Then, with the tools presented in Lecture 7.1.2, after some tensor algebra and equating 4×4 matrices:

$\rho = \rho_A \otimes \rho_B$.

7.5 Entanglement for Two Spins

Susskind started this Lecture with the definition of the density matrix and the following exercise. A warming-up.

Exercise 7.4: About the density matrix for a given state vector

Calculate the density matrix for $|\Psi\rangle = \alpha|u\rangle + \beta|d\rangle$.
This state vector represents a single state(page 198). *The density matrix, which corresponds to a single state is the projection operator onto that state*(page 198).
So, $\rho = |\Psi\rangle\langle\Psi|$,
and the matrix representation is illustrated by the example:
$\rho_{du} = \langle u|\rho|d\rangle = \langle u|\Psi\rangle\langle\Psi|d\rangle$, page 202.
There are four of these elements. The example shown gives with the inner products of the state vector and the basis vectors:
$\rho_{du} = \langle u|(\alpha|u\rangle + \beta|d\rangle)(\alpha^*\langle u| + \beta^*\langle d|)|d\rangle = \beta^*\alpha$.
This one of the four elements of the density matrix.
Or, using the wave function notation, $\rho_{du} = \psi^*(d)\psi(u)$, with subscripts: $\psi_d^*\psi_u$.
Here, use have been made of $\rho_{a'a} = \psi^*(a')\psi(a)$.
Another one:
$\rho_{uu} = \langle u|(\alpha|u\rangle + \beta|d\rangle)(\alpha^*\langle u| + \beta^*\langle d|)|u\rangle = \alpha^*\alpha$.
Then, we finally obtain for the elements $\rho_{a'a}$, of the density matrix ρ:
$\begin{pmatrix} \rho_{uu} & \rho_{ud} \\ \rho_{du} & \rho_{dd} \end{pmatrix} = \begin{pmatrix} \alpha^*\alpha & \alpha^*\beta \\ \beta^*\alpha & \beta^*\beta \end{pmatrix}$.
With the normalization condition
$\langle\Psi|\Psi\rangle = \alpha^*\alpha + \beta^*\beta = 1$,
and the equal probability for up and $down$, $\alpha^*\alpha$ and $\beta^*\beta$ must both be equal to $\frac{1}{2}$ (page 41).
Hence, we find $\alpha = \frac{1}{\sqrt{2}}$ and $\beta = \pm\frac{1}{\sqrt{2}}$. And we find again the two state vectors $|r\rangle$ and $|l\rangle$.
On the other hand, we could have worked with the column representation:
$|\Psi\rangle = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}\alpha\\\beta\end{pmatrix}$ and $\langle\Psi| = (\alpha^* \quad \beta^*)$.
Calculate $|\Psi\rangle\langle\Psi|$:
$|\Psi\rangle\langle\Psi| = \begin{pmatrix}\alpha\\\beta\end{pmatrix}(\alpha^* \quad \beta^*) = \begin{pmatrix}\alpha\alpha^* & \alpha\beta^*\\ \beta\alpha^* & \beta\beta^*\end{pmatrix} = \begin{pmatrix}\alpha^*\alpha & \alpha^*\beta\\ \beta^*\alpha & \beta^*\beta\end{pmatrix}$, Hermitian.

In Eq.(7.18), Susskind summarized how the operator works to obtain one observable of a composite system:

$L_{a'b',ab} = L_{a'a}\delta_{b'b}$.

The left-hand side of this expression is a 4×4 matrix, the right-hand side I a tensor product of two 2×2 matrices.
Note: compare (7.16) and (7.19): Susskind mixed the lower case ψ with the uppercase Ψ.

Compare (7.21) with (7.14). Note: Susskind changed the order of tensor multiplication. It does not matter. See the remark of Susskind just above exercise 7.5, page 209.

Remark: On page 207 Susskind presents some properties of density matrices:

- Density matrices are Hermitian. In general, a density matrix is a sum of projection operators, a linear combination of matrices, each multiplied by its probability, a number. Projection matrices are Hermitian, consequently the density matrix is Hermitian.
- The trace of a density matrix is 1:
  $Tr(\rho) = 1.$
  $P(a)$ is the probability that Alice's system will be left in state a if a measurement is performed. Now,
  $P(a) = \sum_b \psi^*(a,b)\psi(a,b) = \rho_{aa}$.
  Since $\sum_a P(a) = 1, Tr(\rho) = 1.$
- The eigenvalues of the density matrix are all positive and lie between 0 and 1. It follows that if any eigenvalue is 1, all the others are 0. For example:
  $\rho = P_1|\Psi\rangle\langle\Psi| + P_2|\Phi\rangle\langle\Phi|$,
  then,
  $\rho|\Psi\rangle = P_1|\Psi\rangle$, and $\rho|\Phi\rangle = P_2|\Phi\rangle$.
  With $P_1 + P_2 = 1$, we have $0 < P_1 < 1$, and $0 < P_2 < 1$. When $P_1 = 1$, $\rho = |\Psi\rangle\langle\Psi|$, and consequently a pure state. On pages 215 and 216 Susskind explains this in more detail.
- For a pure state $\rho^2 = \rho$ and $Tr(\rho^2) = 1$. A pure state: $\rho = |\Psi\rangle\langle\Psi|$, and
  $\rho^2 = (|\Psi\rangle\langle\Psi|)^2$.
  See page 195:
  $\rho^2 = |\Psi\rangle\langle\Psi|\Psi\rangle\langle\Psi| = |\Psi\rangle\langle\Psi|$.
  Since $Tr(\rho) = 1 \rightarrow Tr(\rho^2) = 1$.
- For a mixed or entangled state $\rho^2 \neq \rho$ and $Tr(\rho^2) < 1$.
  Here mixed or entangled has the same meaning?
  A mixed state, a composite state, an entangled state, ……. A conundrum of states.
  Some explanation could be helpful.
  Well, a subsystem of an entangled state is considered a mixed state. The entangled state is a pure state?
  Some explanation is given on page 208.

Exercise 7.5 About the density matrix and the trace

a) Show that $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^2 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix}$, QED.
It is about matrix multiplication.
b) Now, suppose
$\rho = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$.
Calculate $\rho^2 = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{4}{9} \end{pmatrix}$, then $Tr(\rho) = 1$ and $Tr(\rho^2) = \frac{5}{9}$.
c) ρ, represents a mixed or entangled state one of the properties of density matrices.

Exercise 7.6 For a density matrix the trace equals 1

Use Eq. 7.22 to show that if ρ is a density matrix, then $Tr(\rho) = 1$.
Eq. (7.22) reads: $P(a) = \rho_{aa}$.
$P(a)$ is a diagonal entry of the density matrix ρ. We know $\sum_a P(a) = 1$. Consequently, $Tr(\rho) = 1$. See also the second bullet point above.

7.6 A Concrete Example: Calculating Alice's Density Matrix

In this section Susskind calculates Alice's density matrix.

Remarks: concerning pages 210 and 211.
The calculation of the matrix elements of Alice's density matrix are based on Eq.(7.20) { or for that matter, Eq. (7.23)}:
$\rho_{a'a} = \sum_b \psi^*(a,b)\psi(a',b)$.
Compare this with Eq.(7.17) and you will notice that a' and a have been switched. Well, that's not important. However, using Eq.(7.20) with for example $a' = u, a = d$ and summing over b, you will find:
$\rho_{ud} = \psi^*(d,u)\psi(u,u) + \psi^*(d,d)\psi(u,d) \neq \rho_{ud}$ {from (7.17)}.
I noticed from the matrix element at the bottom of page 211:
$\rho_{ud} = \rho^*_{du}$,
illustrating the density matrix to be Hermitian→ $\rho_{aa'} = \rho^*_{a'a}$.
The same example for $a' = u, a = d$ and summing over b in Eq.(7.17), you obtain a value for $\rho_{du} \neq \rho_{du}$. The latter found at the bottom of page 211.
What to do? An expression for the elements of Alice's density matrix that works, read: $\rho_{a'a} = \sum_b \psi^*(a',b)\psi(a,b)$, Eq.(7.17). Whether or not this is the correct expression does not bother us since the density matrix is Hermitian.
For the concrete example I assume the expressions at the bottom of page 211 to be correct.
At the bottom of page 201 Susskind made the remark: " *There's a subtle point about our notation …….*". And on page 205: "*As I warned, there are lots of indices"*. Subtle indeed. Caveat Subscriptum.
Additionally, you might wonder why Susskind changed for the notation of the wave function from, say, ψ_{ud} to $\psi(u,d)$. I do not know. Getting used to various conventions?

Exercise 7.7 The square of the density matrix

Use Eq. (7.24), $\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$, to calculate ρ^2.

$$\rho^2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}.$$

For a mixed or entangled state, top page 208, $\rho^2 \neq \rho$, and $Tr\ \rho^2 = \frac{1}{2} < 1$.

Remark: With this Exercise 7.7 there is no problem related with the index's conundrum. However, with the following exercise we are not so sure.

So let's return to the full equation for the combined system Eq. (7.15) where we will use the wave equation notation and the subscripts. Also using the subtle point of Susskind on indices at the bottom of page 201, we have the following convention for the elements of matrices- see Eq.(7.1): $L_{aa'} = \langle a|\boldsymbol{L}|a'\rangle$.

The full equation for the expectation value of $\boldsymbol{L}$ is, Eq.(7.15):

$\langle\Psi|\boldsymbol{L}|\Psi\rangle = \sum_{ab,a'b'} \psi^*_{a'b'} L_{ab,a'b'} \psi_{ab}$.

Now, when we use the same notation convention for Alice's density matrix and for Bob's, we have:

Alice $\rho_{aa'} = \sum_b \psi^*_{a'b}\psi_{ab}$, and Bob $\rho_{bb'} = \sum_a \psi^*_{ab'}\psi_{ab}$. **(L7.1)**

Note: when an index occurs twice in the summation the Einstein summation index convention teaches us that we can delete the summation sign **Σ.**

So Eq. (7.17) represents the elements of Alice's matrix. From this we may conclude that ρ_{ud} and ρ_{du} have been switched with no detrimental effect in Exercise 7.7 and at the bottom of page 211. The density matrix is Hermitian.

Exercise 7.8 Calculate A's density matrix and B's density matrix for three given states.

For each of the following states calculate Alice's density matrix and Bob's density matrix. Check their properties.

1) $|\Psi_1\rangle = \frac{1}{2}(|uu\rangle + |ud\rangle + |du\rangle + |dd\rangle)$, and $\psi_{uu} = \psi_{ud} = \psi_{du} = \psi_{dd} = \frac{1}{2}$.

The normalization condition is fulfilled. Now, we use Eq. **L7.1**.

Elements of Alice's density matrix are: $\rho_{uu} = \psi^*_{uu}\psi_{uu} + \psi^*_{ud}\psi_{ud} = \frac{1}{2}$, and this value is found for all the matrix elements of Alice.

So $\rho_{Alice} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$; $Tr\ \rho_{Alice} = 1$ and $\rho^2_{Alice} = \rho_{Alice}$.

Now Bob's matrix. We expect this matrix to be equal to Alice's since $|\Psi_1\rangle$ represents a product matrix with amplitudes $\frac{1}{\sqrt{2}}$. Let's find out. We take ρ_{ud} and find

$\rho_{ud} = \psi^*_{uu}\psi_{ud} + \psi^*_{du}\psi_{dd} = \frac{1}{2}$. This value is found for all the matrix elements of Bob.

So $\rho_{Bob} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$; $Tr\ \rho_{Bob} = 1$ and $\rho^2_{Bob} = \rho_{Bob}$.

The state is not entangled, a product state. See Eq.(6.5).

What does the projection operator look like? On page 201 Susskind writes: *"Indeed, only in the special case of a product state will ρ have the form of a projection operator. In other words, even though the composite system is described by a perfectly pure state, subsystem A must be described by a mixed state".* So for the above product state we have:

$|product\ state\rangle = |\Psi_1\rangle = \left\{\frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)\right\} \otimes \left\{\frac{1}{\sqrt{2}}(|u\rangle + |d\rangle)\right\} = |\Psi_{1Alice}\rangle \otimes |\Psi_{1Bob}\rangle$.

And we can formulate the product state in terms of a projection operator.

$\rho_{product\ state} = |\Psi_1\rangle\langle\Psi_1|$. With the column vector representation of $|\Psi_1\rangle$ and elements: $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$, we obtain for the 4×4 matrix representation of $\rho_{product\ state} = \begin{pmatrix} \frac{1}{4} & \cdots & \frac{1}{4} \\ \vdots & \ddots & \vdots \\ \frac{1}{4} & \cdots & \frac{1}{4} \end{pmatrix}$.

$Tr\ \rho_{product\ state} = 1$, and $Tr\ \rho^2_{product\ state} = 1$.

In addition: $\rho_{product\ state} = |\Psi_1\rangle\langle\Psi_1| = \rho_{Alice} \otimes \rho_{Bob}$.

2) $|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|uu\rangle + |dd\rangle)$. A triplet state $|T_2\rangle$, certainly not a product state as we learned in Lecture 6. For this case the amplitudes are: $\psi_{uu} = \psi_{dd} = \frac{1}{\sqrt{2}}$, and $\psi_{ud} = \psi_{du} = 0$.

The elements of Alice's matrix are found with help of Eq. **(L7.1)**:

$\rho_{Alice} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $\rho^2_{Alice} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$; So $\rho_{Alice} \neq \rho^2_{Alice}$ and $Tr\ \rho^2_{Alice} = \frac{1}{2} < 1$.

The elements of Bob's matrix are also found with help of Eq. **(L7.1):**

$\rho_{Bob} = \rho_{Alice} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $\rho^2_{Bob} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$; So $\rho_{Bob} \neq \rho^2_{Bob}$ and $Tr\ \rho^2_{Bob} = \frac{1}{2} < 1$.

Alice's and Bob's density matrices are proportional to the unit matrix. In section 7.7.2 Susskind will explain this phenomena.

Here it is shown again the triplet state $|T_2\rangle$ to be an entangled state. Is this called a single state?

What does the projection operator look like? Well, certainly it is a 4×4 matrix of which the elements are found by the vector representation of $\rho = |\Psi\rangle\langle\Psi|$. What is the meaning of this projection operator?

The elements of $|\Psi\rangle$ are: $0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$.

The matrix then reads: $\rho_{entangled} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

So $Tr\ \rho_{entangled} = 1$ and $Tr\ \rho^2_{entangled} = 1$.

Furthermore $\rho_{entangled} = |\Psi_2\rangle\langle\Psi_2| \neq \rho_{Alice} \otimes \rho_{Bob}$.

3) $|\Psi_3\rangle = \frac{1}{5}(3|uu\rangle + 4|ud\rangle$. What kind of state is this one? We'll find out. It is normalized and looks like meeting Pythagoras.
The amplitudes are $\psi_{uu} = \frac{3}{5}, \psi_{ud} = \frac{4}{5}$, and $\psi_{du} = \psi_{dd} = 0$.
The elements of Alice's matrix are found with help of Eq. **(L7.1)**:
$\rho_{Alice} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\rho_{Alice}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; $\rho_{Alice}^2 = \rho_{Alice}$ and $Tr\ \rho_{Alice} = Tr\ \rho_{Alice}^2 = 1$.
The elements of Bob's matrix are also found with help of Eq. **(L7.1):**
$\rho_{Bob} = \frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$ and $\rho_{Bob}^2 = \frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}$; $\rho_{Bob}^2 = \rho_{Bob}$ and $Tr\ \rho_{Bob} = Tr\ \rho_{Bob}^2 = 1$.
What kind of state is $|\Psi_3\rangle$? Looking at the density matrices of Alice and Bob, this state is a product state.
Using Eq.(6.4), normalisation conditions, and Eq.(6.5), the expanded product state and composite notation:
$|product\ state\rangle = |u\rangle \otimes \frac{1}{5}(3|u\rangle + 4|d\rangle) = |\Psi_{3Alice}\rangle \otimes |\Psi_{3Bob}\rangle$.
$\rho_{product\ state} = |\Psi_3\rangle\langle\Psi_3|$. With the vector representation of $|\Psi_3\rangle$ and elements: $\frac{3}{5}, \frac{4}{5}, 0, 0$, we obtain for the 4×4 matrix representation of

$$\rho_{product\ state} = \frac{1}{25}\begin{pmatrix} 9 & 12 & 0 & 0 \\ 12 & 16 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$Tr\ \rho_{product\ state} = 1$, and $Tr\ \rho_{product\ state}^2 = 1$.
In addition $\rho_{product\ state} = |\Psi_1\rangle\langle\Psi_1| = \rho_{Alice} \otimes \rho_{Bob}$.

Remark: Density matrix of a product state equals the tensor product of the two constituting states

In the above Exercise 7.8 we included the density matrix of the composite state and found for the product state the density matrix, $\rho_{product\ state}$, to be the tensor product of the two building blocks: Alice's and Bob's state, $\rho_{Alice} \otimes \rho_{Bob}$. Can we prove that without using amplitudes? Let's have a look.
Proof by the method of contradiction:
The state vector for the product state is $|\Psi\rangle$,for Alice's $|\Psi_A\rangle$ and Bob's $|\Psi_B\rangle$. So we want to prove: $|\Psi\rangle\langle\Psi| = |\Psi_A\rangle\langle\Psi_A| \otimes |\Psi_B\rangle\langle\Psi_B|$, **(L7.2)**
or $\rho_{product\ state} = \rho_{Alice} \otimes \rho_{Bob}$.
We know for a ket vector:
$|\Psi_i\rangle\langle\Psi_i|\ |\Psi_i\rangle = |\Psi_i\rangle$, **(L7.3)**
where i refers to Alice's or Bob's state vector: Ψ_A or Ψ_B respectively. (see page 194, second bullet point).
Eq. **(L7.3)** applies also to the bra vectors $\langle\Psi_i|$.
Now, plug into Eq. **(L7.2)** in the left part and the right part on the right-hand side $|\Psi_B\rangle$:
$|\Psi\rangle\langle\Psi||\Psi_B\rangle = |\Psi_A\rangle\langle\Psi_A| \otimes |\Psi_B\rangle\langle\Psi_B|\ |\Psi_B\rangle$, this gives with Eq. **(L7.3)**
$|\Psi\rangle\langle\Psi||\Psi_B\rangle = |\Psi_A\rangle\langle\Psi_A| \otimes |\Psi_B\rangle$.
Plug into the right part and the left part on the left-hand side of the last equation $\langle\Psi_A|$:
$\langle\Psi_A|\ |\Psi\rangle\langle\Psi||\Psi_B\rangle = \langle\Psi_A|\ |\Psi_A\rangle\langle\Psi_A| \otimes |\Psi_B\rangle$, this gives with Eq. **(L7.3)**
$\langle\Psi_A|\ |\Psi\rangle\langle\Psi|\Psi_B\rangle = \langle\Psi_A| \otimes |\Psi_B\rangle$.
Rewrite this equation by reversing the order of the two numbers on the left-hand side:
$\langle\Psi|\Psi_B\rangle\langle\Psi_A||\Psi\rangle = \langle\Psi_A| \otimes |\Psi_B\rangle$. In this equation we substitute for $|\Psi\rangle$, $|\Psi_A\rangle \otimes |\Psi_B\rangle$, and for $\langle\Psi|$, $\langle\Psi_A| \otimes \langle\Psi_B|$. Then we find:

$\langle\Psi_A| \otimes \langle\Psi_B||\Psi_B\rangle\langle\Psi_A||\Psi_A\rangle \otimes |\Psi_B\rangle = \langle\Psi_A| \otimes |\Psi_B\rangle$ or the left hand side equals the right hand side:
left $\langle\Psi_A| \otimes |\Psi_B\rangle = \langle\Psi_A| \otimes |\Psi_B\rangle$ right.
There is no contradiction and we have proven both product state examples in Exercise 7.8.

7.7 Tests for Entanglement

In this Lecture answers are looked for to questions like: *are there various degrees of entanglement, can they be quantified,* etc.
Entanglement is the quantum mechanical generalization of correlation.
It is about a mathematical procedure.

7.7.1 The Correlation Test for Entanglement

It is about expectation values.

Exercise 7.9 For a product state the correlation to be zero

Given any Alice observable $\boldsymbol{A}$ and any Bob observable $\boldsymbol{B}$**,** show that for a product state, the correlation $C(A,B)$ is zero.
Note: A product state is a state with zero entanglement and consequently the correlation is zero. See Exercise 6.1.
We know, Eq. (7.13): $\langle\boldsymbol{A}\rangle = Tr\ \rho_A\boldsymbol{A}$, $\langle\boldsymbol{B}\rangle = Tr\ \rho_B\boldsymbol{B}$, and $\langle\boldsymbol{AB}\rangle = Tr\ \rho_{AB}\boldsymbol{AB}$**.**
We have a product state. So, we can make use of $\rho_{AB} = \rho_{product\ state} = \rho_{Alice} \otimes \rho_{Bob}$. Furthermore, the trace of a product of two matrices does not depend on their order of multiplication. Then, we can write for the correlation, Eq.(7.13) the expression for the expectation value:
$C(A,B) = Tr\ \boldsymbol{AB}\rho_A \otimes \rho_B - Tr\ \boldsymbol{A}\rho_A\ Tr\ \boldsymbol{B}\rho_B = 0,$
and since $\boldsymbol{A}$ operates on Alice's state vector and $\boldsymbol{B}$ on Bob's state vector
$C(A,B) = Tr\ \boldsymbol{A}\rho_A \otimes\ \boldsymbol{B}\rho_B - (\ Tr\ \boldsymbol{A}\rho_A)\ (Tr\ \boldsymbol{B}\rho_B) = 0.$
Now, plug into this equation a general 2×2 matrix for
$\boldsymbol{A}\rho_A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$
and for
$\boldsymbol{B}\rho_B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$
Then
$Tr\ \boldsymbol{A}\rho_A \otimes\ \boldsymbol{B}\rho_B = a_{11}b_{11} + a_{11}b_{22} + a_{22}b_{11} + a_{22}b_{22},$
and
$(\ Tr\ \boldsymbol{A}\rho_A)(Tr\ \boldsymbol{B}\rho_B) = a_{11}b_{11} + a_{11}b_{22} + a_{22}b_{11} + a_{22}b_{22}.$
Hence,
$C(A,B) = Tr\ \boldsymbol{A}\rho_A \otimes\ \boldsymbol{B}\rho_B - (\ Tr\ \boldsymbol{A}\rho_A)\ (Tr\ \boldsymbol{B}\rho_B) = 0.$
For a product state there is no correlation.
$C(A,B) = \langle\boldsymbol{AB}\rangle - \langle\boldsymbol{A}\rangle\langle\boldsymbol{B}\rangle = 0$

Then, Susskind writes: *"From this exercise we ca learn something about entanglement. If a system is in state where one can find any two observables* $\boldsymbol{A}$ *and* $\boldsymbol{B}$ *that are correlated-meaning that* $C(A,B) \neq 0$ *– then the state is entangled."* Well, what I learned from this exercise for a product state $C(A,B) = 0$. Obviously, when $C(A,B) \neq 0$ is defined for an entangled state, the answer is almost trivial.

7.7.2 The Density Matrix Test for Entanglement

In this Lecture Susskind described a test for entanglement where only one of the density matrices, A's one or B's one, is needed.
Here again, the proof is based on a product state→being not entangled.
On page 217, in the middle,
"…, *she finds something very disappointing: the density matrix is proportional to the unit matrix. All the eigenvalues are equal, and given that they all sum up to unity, each eigenvalue is equal to* $1/N_A$." N_A is the dimension of Alice's space of states. See Exercise 7.8, **2)**.

7.8 The Process of Measurement

In the introduction of this Lecture Susskind mentioned the debates on *so-called reality.* An interesting remark. See Smolin (1).
Then, Susskind mentioned: *"I'm going to steer away from those debates and stick to the facts."*
Whatever facts mean with respect to quantum mechanics.
Measurement and quantum mechanics: *a measurement involves a system and an apparatus.*
Then, Susskind constructed a mathematical system including the measurement equipment, pages 219-221.
Remark:
At the top of page 220, six basis vectors are presented for the composite (tensor product) space of states. So, a state is represented by a column vector of six components? In the remark below, I presented the final state vector to be 2-dimensional. Does a 6-dimensional vector collapse into a 2-dimensional one?

Exercise 7.10: a state vector→completely unentangled state

Verify that the state vector, including the measurement equipment in Eq.(7.30), represents a completely unentangled state.
The initial state vector is:
$|\Psi\rangle = \alpha_u|u,b\rangle + \alpha_d|d,b\rangle$, Eq.(7.30) **.**
Unentanglement means, in general, for the wave function, $\psi_{ab} = \psi_a\psi_b$.
For the above state vector, after inspection, we have: $\psi_{ub} = \alpha_u$, and $\psi_{db} = \alpha_d$.
The initial spin state $|\Psi_s\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$, and for the apparatus: $|\Psi_{ap}\rangle = |b\rangle$.
Verification: the initial state vector can be written as: $|\Psi\rangle = (\alpha_u|u\rangle + \alpha_d|d\rangle) \otimes |b\rangle = \alpha_u|u,b\rangle + \alpha_d|d,b\rangle$.
Comparing this with Eq.(6.5), we know $|\Psi\rangle$ to represent a product state.

Remark:
On page 221 Susskind showed the final state(fs) : $|\Psi_{fs}\rangle$ after the measurement.
For the up state: $|u,b\rangle \rightarrow |u,+1\rangle$.
If the spin is in the $down$ state: $|d,b\rangle \rightarrow |d,-1\rangle$.
The state $|\Psi\rangle$ changes from $\alpha_u|u,b\rangle + \alpha_d|d,b\rangle$ into the final state:
$|\Psi_{fs}\rangle = \alpha_u|u,+1\rangle + \alpha_d|d,-1\rangle$ **.**
In column vector representation:
$|\Psi_{fs}\rangle = \alpha_u \begin{pmatrix}1\\0\end{pmatrix} \otimes (+1) + \alpha_d \begin{pmatrix}0\\1\end{pmatrix} \otimes (-1) = \begin{pmatrix}\alpha_u\\-\alpha_d\end{pmatrix}$.
This is an entangled state . This final state $|\Psi_{fs}\rangle$ cannot be written as
$(\alpha_u|u\rangle + \alpha_d|d\rangle) \otimes (|+1\rangle + |-1\rangle) = \alpha_u|u,+1\rangle + \alpha_u|u,-1\rangle + \alpha_d|d,+1\rangle + \alpha_d|d-1\rangle$ **.**

Since we find four contradictory expressions: $\alpha_u = 1, \alpha_u = 0, \alpha_d = 0$, and $\alpha_d = 1$, by comparing this tensor product with the final entangled state $|\Psi_{fs}\rangle$.
Conclusion, the wave function ψ_{ab} cannot be factorized into $\psi_a \psi_b$.
Susskind added: "*In fact, if $\alpha_u = -\alpha_d$, it is the maximally entangled singlet state*".
Let us have a look. For the final state we find with Eq. **(L7.1):**

$$\rho_s = \begin{pmatrix} \alpha_u^* \alpha_u & 0 \\ 0 & \alpha_d^* \alpha_d \end{pmatrix},$$

and

$Tr\ \rho_s = \alpha_u^* \alpha_u + \alpha_d^* \alpha_d$.

Furthermore with Eq. **(L7.1)** for the apparatus

$Tr\ \rho_{app} = \alpha_u^* \alpha_u + \alpha_d^* \alpha_d$.

Normalization of the final state $|\Psi_{fs}\rangle$ shows $\alpha_u^* \alpha_u + \alpha_d^* \alpha_d = 1$.
With help of this expression, we have $Tr\ \rho_s^2 \neq 1$, and $Tr\ \rho_{app}^2 \neq 1$. The density matrix ρ_{fs} of the final state is found with help of the column vector representation of $|\Psi_{fs}\rangle$. The elements of the column are: $\alpha_u, -\alpha_d$.
ρ_{fs} is a 2× 2 matrix:

$$\begin{pmatrix} \alpha_u^* \alpha_u & \alpha_d^* \alpha_u \\ \alpha_u^* \alpha_d & \alpha_d^* \alpha_d \end{pmatrix},$$

and

$Tr\ \rho_{fs} = \alpha_u^* \alpha_u + \alpha_d^* \alpha_d = 1$. In addition, $Tr\ \rho_{fs}^2 = (\alpha_u^* \alpha_u + \alpha_d^* \alpha_d)^2 = 1$. A pure state? Last bullet point on page 207.

With $\alpha_u = -\alpha_d$ we find $\alpha_u^* \alpha_u = \alpha_d^* \alpha_d$ and $\alpha_u^* \alpha_u = \frac{1}{2}$. Neglecting a phase factor this results into $\alpha_u = \frac{1}{\sqrt{2}}$. Now it becomes clear that for $\alpha_u = -\alpha_d$ there is maximum entanglement.
Look at ρ_s:
$\rho_s = \begin{pmatrix} \alpha_u^* \alpha_u & 0 \\ 0 & \alpha_d^* \alpha_d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The density matrix is proportional to the unit matrix. Each measurement outcome is equally likely.
I did not understand the line at the bottom of page 222: "*But then there is good old Charlie….*".
On the pages 232, 233 and 234 Susskind summarizes entanglement for a product state, a singlet state and a "near singlet". He concludes this lecture with a remark on reality and confusion. So, the debate, mentioned at the beginning of this lecture, continues.

7.9 Entanglement and Locality.

This Lecture starts with another debate: is quantum mechanics local or nonlocal?
Then, Susskind explained the meaning of locality.

7.10 The Quantum Sim: An Introduction to Bell's Theorem

This Lecture started with the remark: *"… that unitary played a prominent role in guaranteeing that no signal can be send instantaneously."*
Susskind spent attention to the subject matter: *"… he (Einstein) and Bell were talking about a totally different notion of locality…."* , compared to the locality discussed in Lecture 7.9.
Then, Susskind invites you to play a computer game. Starting with one computer and next,

with two computers.
Susskind concluded this Lecture with the remark the problem not being a quantum mechanical problem.[3]

7.11 Entanglement Summary.

This Lecture started with the remark entanglement being *"the hardest concept to accept".* Then, a compact summary of entanglement is presented.

Rap Sheet 1 State-Vector No Entanglement

This rap sheet is about a product state (No entanglement).
Wanted for: Excessive Locality, Impersonating a Classical system.
Description: Each subsystem is fully characterized. There are no correlations between A's and B's systems.
The numerical values for normalization:
A product state $|\Psi_{Product}\rangle$ is the result of completely independent preparations by A's - and B's system. See page 163:
$|\Psi_A\rangle = \alpha_u|u\rangle + \alpha_d|d\rangle$, and
$|\Psi_B\rangle = \beta_u|u\rangle + \beta_d|d\rangle$.
So, the State-Vector
$|\Psi_{Product}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, Eq.(6.5).
$\langle\Psi_A|\Psi_A\rangle = 1$ and $\langle\Psi_B|\Psi_B\rangle = 1$.
These normalization conditions produce Eq.(6.4):
$\alpha_u^*\alpha_u + \alpha_d^*\alpha_d = 1$ and $\beta_u^*\beta_u + \beta_d^*\beta_d = 1$.
Before going any further, let's do some ket, bra and tensor algebra.
$\langle\Psi_{Product}|\Psi_{Product}\rangle = (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)(\beta_u^*\beta_u + \beta_d^*\beta_d) = \langle\Psi_A|\Psi_A\rangle\langle\Psi_B|\Psi_B\rangle = 1.$
Also $\langle\Psi_{Product}|\Psi_{Product}\rangle = (\langle\Psi_A| \otimes \langle\Psi_B|)(|\Psi_A\rangle \otimes |\Psi_B\rangle) = \langle\Psi_A| \otimes \langle\Psi_B|\Psi_A\rangle \otimes |\Psi_B\rangle$.
Now I want to recall the footnote of Susskind on page 193: "*Sometimes we can change left-to right ordering as well, but that requires more care*".
So can we write:
$\langle\Psi_{Product}|\Psi_{Product}\rangle = (\langle\Psi_A| \otimes \langle\Psi_B|)(|\Psi_A\rangle \otimes |\Psi_B\rangle) = \langle\Psi_A| \otimes |\Psi_A\rangle\langle\Psi_B| \otimes |\Psi_B\rangle$?
Does it appear that $\langle\Psi_A| \otimes |\Psi_A\rangle = \langle\Psi_A|\Psi_A\rangle = 1$, and $\langle\Psi_B| \otimes |\Psi_B\rangle = \langle\Psi_B|\Psi_B\rangle = 1$?
When we use the column vector representation for $|\Psi_A\rangle$and $|\Psi_B\rangle$: it seems to be all right.
This is nice!
The density matrix for the product state:
with **Eq.(L7.1)** and the product state vector $|\Psi_{Product}\rangle$ we find for Alice's density matrix:
$\begin{pmatrix} \alpha_u^*\alpha_u & \alpha_d^*\alpha_u \\ \alpha_u^*\alpha_d & \alpha_d^*\alpha_d \end{pmatrix}$. The determinant for the eigenvalue λ is:
$\begin{vmatrix} \alpha_u^*\alpha_u - \lambda & \alpha_d^*\alpha_u \\ \alpha_u^*\alpha_d & \alpha_d^*\alpha_d - \lambda \end{vmatrix} = 0$. Which produces a quadratic equation for λ:
$\lambda\big(\lambda - (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)\big) = 0,$
where use has been made of $\alpha_u^*\alpha_u\alpha_d^*\alpha_d - \alpha_d^*\alpha_u\alpha_u^*\alpha_d = 0$.
The one nonzero eigenvalue: $\lambda = 1$.

[3] An example of entanglement research is presented in the Science and Technology Section of The Economist August 15th 2020. The title: Quantum Mechanics, A flutter in time. "*There are no butterflies in the quantum realm.*"

The associated eigenvector, with general components (a_1, a_2), is found in the following way:

$\begin{pmatrix} \alpha_u^*\alpha_u & \alpha_d^*\alpha_u \\ \alpha_u^*\alpha_d & \alpha_d^*\alpha_d \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \lambda\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} \alpha_u \\ \alpha_d \end{pmatrix}.$

As Susskind says: "*The same goes for Bob*".

The wave function can be factorized.

This follows from comparison of the composite notation for the product state in Eq.(6.5):

$|\Psi_{product}\rangle = \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle,$

with the most general vector in the composite space of states on page 165:

$|\Psi\rangle = \psi_{uu}|uu\rangle + \psi_{ud}|ud\rangle + \psi_{du}|du\rangle + \psi_{dd}|dd\rangle.$

So, for example $\psi_{ud} = \alpha_u\beta_d$.

Another approach: let's take ψ_{ud}

$\psi_{ud} = \langle ud|\Psi_{product}\rangle = \langle u| \otimes \langle d|\Psi_A\rangle \otimes |\Psi_B\rangle = \alpha_u\langle d|\Psi_B\rangle = \alpha_u\beta_d.$

Take notice that use has been made of $\langle u|$ multiplies with $|\Psi_A\rangle$ and $\langle d|$ with $|\Psi_B\rangle$. To find out about this tensor multiplication, use the column vector presentation of $|ud\rangle$ and

$$|\Psi_{product}\rangle : \langle ud|\Psi_{product}\rangle = (0\ 1\ 0\ 0)\cdot\begin{pmatrix} \alpha_u\beta_u \\ \alpha_u\beta_d \\ \alpha_d\beta_u \\ \alpha_d\beta_d \end{pmatrix} = \alpha_u\beta_d.$$

Expectation values:

$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = 1.$

We can prove this equality by straightforward applying the definition of $\langle\sigma_x\rangle$, $\langle\sigma_y\rangle$ and $\langle\sigma_z\rangle$:

$\langle\sigma_x\rangle = \langle\Psi_{product}\ |\sigma_x|\Psi_{product}\rangle$, etc.

With $|\Psi\rangle = \psi_{uu}|uu\rangle + \psi_{ud}|ud\rangle + \psi_{du}|du\rangle + \psi_{dd}|dd\rangle$, the Pauli matrices and the Kronecker delta we find:

$\langle\sigma_x\rangle = \alpha_d^*\alpha_u + \alpha_u^*\alpha_d$, $\langle\sigma_y\rangle = i(\alpha_d^*\alpha_u - \alpha_u^*\alpha_d)$, and $\langle\sigma_z\rangle = \alpha_u^*\alpha_u - \alpha_d^*\alpha_d$.

This results into:

$\langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2 + \langle\sigma_z\rangle^2 = (\alpha_u^*\alpha_u + \alpha_d^*\alpha_d)^2 = 1.$ (Eq. 3.27).

The same goes for Bob's τ with α replaced by β.

Another approach for expectation values is using the density matrix ρ. We take for example $\langle\sigma_x\rangle$.

We already calculated the density matrix $\rho_{Alice} = \begin{pmatrix} \alpha_u^*\alpha_u & \alpha_d^*\alpha_u \\ \alpha_u^*\alpha_d & \alpha_d^*\alpha_d \end{pmatrix}$.

So,

$$\langle\sigma_x\rangle = Tr\ \rho_{Alice}\sigma_x = Tr\begin{pmatrix} \alpha_u^*\alpha_u & \alpha_d^*\alpha_u \\ \alpha_u^*\alpha_d & \alpha_d^*\alpha_d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \alpha_d^*\alpha_u + \alpha_u^*\alpha_d.$$

Correlation: $\langle\sigma_z\tau_z\rangle - \langle\sigma_z\rangle\langle\tau_z\rangle = 0.$

With the knowledge of $\langle\sigma_z\rangle = \alpha_u^*\alpha_u - \alpha_d^*\alpha_u$ we find:

$\langle\tau_z\rangle = \beta_u^*\beta_u - \beta_d^*\beta_d.$

This gives for $\langle\sigma_z\rangle\langle\tau_z\rangle = (\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)(\beta_u^*\beta_u - \beta_d^*\beta_d).$

$\langle\sigma_z\tau_z\rangle = \langle\Psi_{product}|\sigma_z\tau_z|\Psi_{product}\rangle = (\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)(\beta_u^*\beta_u - \beta_d^*\beta_d) = \langle\sigma_z\rangle\langle\tau_z\rangle.$

Hence

$\langle\sigma_z\tau_z\rangle - \langle\sigma_z\rangle\langle\tau_z\rangle = 0,$
no correlation.
Now the approach with the density matrix. I like to recall that for a product state:

$\rho_{product} = \rho_{Alice} \otimes \rho_{Bob}$, and $\sigma_z \otimes \tau_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Now with all the elements known: $\langle\sigma_z\tau_z\rangle = Tr\ \rho_{product}\sigma_z\tau_z =$

$$= Tr\ \rho_{Alice}.\otimes \rho_{Bob} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\alpha_u^*\alpha_u - \alpha_d^*\alpha_d)(\beta_u^*\beta_u - \beta_d^*\beta_d).$$

To find this, you must do a lot of bookkeeping. Let's look again at :
$\langle\sigma_z\tau_z\rangle = Tr\ \rho_{product}\sigma_z\tau_z = Tr\ \rho_{Alice} \otimes \rho_{Bob}\sigma_z \otimes \tau_z.$
We know σ_z to operate on Alice's system and τ_z on Bob's:
$\langle\sigma_z\tau_z\rangle = Tr\ \rho_{product}\sigma_z\tau_z = Tr\ \rho_{Alice}\sigma_z \otimes \rho_{Bob}\tau_z,$ and
$\langle\sigma_z\rangle\langle\tau_z\rangle = Tr\ \rho_{Alice}\sigma_z Tr\ \rho_{Bob}\tau_z$.
We substitute for $\rho_{Alice}\sigma_z$ a general 2×2 matrix A with elements A_{ij} and for $\rho_{Bob}\tau_z$ the 2×2 matrix B with elements B_{kl}. We know $Tr\ A \otimes B = Tr\ A\ Tr\ B$. Then $\langle\sigma_z\tau_z\rangle = A_{ii}B_{kk}$ and $\langle\sigma_z\rangle\langle\tau_z\rangle = A_{ii}B_{kk}$. With Einstein's summation convention:
$\langle\sigma_z\tau_z\rangle - \langle\sigma_z\rangle\langle\tau_z\rangle = 0,$
no correlation.

Rap Sheet 2 State-Vector Maximum Entanglement

This rap sheet is about a singlet state (maximum entanglement).
Wanted for: Nonlocality, Complete Quantum Weirdness.
Description: The complete system is fully characterized. There is no information about A's - or B's subsystems.
The state vector:
$|\Psi_{ent}\rangle = |sing\rangle = \frac{1}{\sqrt{2}}(|ud\rangle - |du\rangle),\ \psi_{uu} = \psi_{dd} = 0$
and in general
$|\Psi\rangle = \psi_{uu}|uu\rangle + \psi_{ud}|ud\rangle + \psi_{du}|du\rangle + \psi_{dd}|dd\rangle.$
So,
$\psi_{ud} = -\psi_{du} = \frac{1}{\sqrt{2}}.$
The normalization condition is: $\psi_{uu}^*\psi_{uu} + \psi_{ud}^*\psi_{ud} + \psi_{du}^*\psi_{du} + \psi_{dd}^* = 1$, page 166.
The density matrix:
For the full composite system $\rho^2 = \rho$, and $Tr(\rho^2)$. Consequently we can ρ_{ent} as a projection operator. Definition of: *the full composite system?*
$\rho_{ent} = |\Psi_{ent}\rangle\langle\Psi_{ent}|$, with the amplitudes of the state vector, $0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0.$
With the column vector representation of the state vector:

$$|\Psi_{ent}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

we obtain for 4×4 density matrix:

$\rho_{ent} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then $\rho_{ent}^2 = \rho_{ent}$.

Note: top page 208, bullet point, *"For a mixed or entangled state $\rho^2 \neq \rho$, and $Tr(\rho^2) < 1$"*? And, $Tr\ \rho_{ent} = Tr\ \rho_{ent}^2 = 1$. What is going on here? A full composite system is not a mixed or entangled system? However, this rap sheet is about maximum entanglement. Is there a difference between entanglement and maximum entanglement?
Page 201: "*Indeed, only in the very special case of a product state will ρ have the form of a projection operator*".

The density matrix of A's subsystem, with help of Eq. (**L7.1):**
$\rho_{aa'} = \sum_b \psi_{a'b}^* \psi_{ab}$, we find for the elements of A's density matrix: $\rho_{uu} = \rho_{dd} = \frac{1}{2}$, and $\rho_{ud} = \rho_{du} = 0$. So,

$\rho_{Alice} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and $\rho_A^2 = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$; $Tr\ \rho_{Alice}^2 < 1$.

$\rho_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The density matrix is proportional to the unit matrix with eigenvalues that add up to 1. Each measurement outcome is equally likely.
Wave function:
$\psi_{ud} = -\psi_{du} = \frac{1}{\sqrt{2}}$, cannot be factorized.
Exercise 6.3: $|sing\rangle$ cannot be written as a product state. Eq. 6.5:
$|\Psi_{product}\rangle = \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$,
represents the product state. We compare this equation with the expression for $|sing\rangle$ and find α_u and/or β_u is zero. Also α_d and/or β_d is zero. Consequently, the state $|sing\rangle$ cannot be written as a product state.
Expectation values:
$\langle\sigma_x\rangle = \langle\sigma_y\rangle = \langle\sigma_z\rangle = 0$, on page 173 and 174 Susskind give the proof of the numerical values of these expectation values. And he concludes: "Needless to say, the same is true for the expectation value of τ".
$\langle\tau_x\sigma_x\rangle = \langle\tau_y\sigma_y\rangle = \langle\tau_z\sigma_z\rangle = -1$, from page 177 and 178 we learned:
$\tau_x\sigma_x|sing\rangle = \tau_y\sigma_y|sing\rangle = \tau_z\sigma_z|sing\rangle = -|sing\rangle$, then
$\langle sing|\tau_x\sigma_x|sing\rangle = \langle\tau_x\sigma_x\rangle = \langle sing|\tau_y\sigma_y|sing\rangle = \langle\tau_y\sigma_y\rangle = \langle sing|\tau_z\sigma_z|sing\rangle = \langle\tau_z\sigma_z\rangle = -\langle sing|sing\rangle = -1$.
Then the correlation $\langle\sigma_z\tau_z\rangle - \langle\sigma_z\rangle\langle\tau_z\rangle = -1$. Reminder: σ_z and τ_z are just one and the same operator.

Rap Sheet 3 State-Vector Partial Entanglement

The near-singlet(ns) state vector (Partial Entanglement).
Incomplete information about the composite system and about the subsystems. There is

some information about the composite system, and some about each subsystem.
The state vector: $|\Psi_{ns}\rangle = \sqrt{0.6}|ud\rangle - \sqrt{0.4}|du\rangle$,
$\psi_{uu} = \psi_{dd} = 0$ and $\psi_{ud} = \sqrt{0.6}$ and $\psi_{du} = -\sqrt{0.4}$.
The normalization condition is: $\psi^*_{uu}\psi_{uu} + \psi^*_{ud}\psi_{ud} + \psi^*_{du}\psi_{du} + \psi^*_{dd} = 1$.
The density matrix:
For the composite system $\rho_{ns} = |\Psi_{ns}\rangle\langle|\Psi_{ns}|$, with amplitudes of the state vector:
$0, \sqrt{0.6}, -\sqrt{0.4}, 0,$

$$|\Psi_{ns}\rangle = \begin{pmatrix} 0 \\ \sqrt{0.6} \\ -\sqrt{0.4} \\ 0 \end{pmatrix},$$

and assuming $\rho_{ns} = |\Psi_{ns}\rangle\langle|\Psi_{ns}|$

$$\rho_{ns} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.6 - \sqrt{0.24} & 0 & \\ 0 - \sqrt{0.24} & 0.4 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then,
$\rho^2_{ns} \neq \rho_{ns}$ and $Tr\ \rho^2_{ns} < 1$.

For A's subsystem, with help of Eq. **(L7.1)**: $\rho_{aa'} = \sum_b \psi^*_{a'b}\psi_{ab}$, and the expansion on page 211, A's density matrix:

$$\rho_A = \begin{pmatrix} 0.6 & 0 \\ 0 & 0.4 \end{pmatrix} \rightarrow Tr\ \rho_A = 1 \text{ and,}$$

$$\rho^2_A = \begin{pmatrix} 0.36 & 0 \\ 0 & 0.16 \end{pmatrix} \rightarrow Tr\ \rho^2_A < 1.$$

Wave function:
Compare the state vector $|\Psi_{ns}\rangle = \sqrt{0.6}|ud\rangle - \sqrt{0.4}|du\rangle$, with the product state vector $|\Psi_{product}\rangle = \alpha_u\beta_u|uu\rangle + \alpha_u\beta_d|ud\rangle + \alpha_d\beta_u|du\rangle + \alpha_d\beta_d|dd\rangle$, and we find α_u and/or β_u is zero. Also β_u and/or β_d is zero. Consequently, the state $|\Psi_{ns}\rangle$ cannot be written as a product state and not be factorized.
Expectation values, with Eq.(7.4)

$$\langle\sigma_z\rangle = \langle\Psi_{ns}|\sigma_z\ |\Psi_{ns}\rangle = \langle\Psi_{ns}|\sigma_z \otimes I\ |\Psi_{ns}\rangle =$$

$$= \left(0, \sqrt{0.6}, -\sqrt{0.4}, 0\right)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ \sqrt{0.6} \\ -\sqrt{0.4} \\ 0 \end{pmatrix} = 0.6 - 0.4 = 0.2\ .$$

$$\langle\sigma_x\rangle = \langle\Psi_{ns}|\sigma_x\ |\Psi_{ns}\rangle = \langle\Psi_{ns}|\sigma_x \otimes I\ |\Psi_{ns}\rangle =$$

$$= \left(0, \sqrt{0.6}, -\sqrt{0.4}, 0\right)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ \sqrt{0.6} \\ -\sqrt{0.4} \\ 0 \end{pmatrix} = 0.$$

Likewise, $\langle\sigma_y\rangle = 0$.
$\langle\tau_z\rangle = \langle\Psi_{ns}|\tau_z\ |\Psi_{ns}\rangle = \langle\Psi_{ns}|I \otimes \tau_z\ |\Psi_{ns}\rangle = -0.2$.
Furthermore $\langle\tau_x\rangle = \langle\tau_y\rangle = 0$.

$$\tau_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

gives with $\rho_{ns} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.6 - \sqrt{0.24} & 0 & 0 \\ 0 & -\sqrt{0.24} & 0.4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$:

$\langle \tau_z \sigma_z \rangle = Tr\ \rho_{ns}\ \tau_z \otimes \sigma_z = -1.$

$$\tau_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

gives with ρ_{ns}:

$\langle \tau_x \sigma_x \rangle = Tr\ \rho_{ns}\ \tau_x \otimes \sigma_x = -2\sqrt{0.24}.$

Or

$\langle \tau_x \sigma_x \rangle = \langle \Psi_{ns} | (I \otimes \tau_x)(\sigma_x \otimes I)\ | \Psi_{ns} \rangle.$

Caveat: $\rho_{ns}\tau_x \otimes \sigma_x$ is not an Hermitian operator!

Correlation:

We have all the building blocks. In addition, $\langle \tau_z \sigma_z \rangle = \langle \sigma_z \tau_z \rangle$. So $\langle \sigma_z \tau_z \rangle - \langle \sigma_z \rangle \langle \tau_z \rangle = -0.96$.

The correlation is between -1 and $+1$, or more accurate $0 < |\langle \tau_z \sigma_z \rangle| < 1$.

So, for a partially entangled state the $0 < |\text{correlation}| < 1$.

Exercises 7.11 and 7.12 are included in the rap sheets.

7.12 Definitions

- *mixed state*, when the density matrix of that state is a mix of several projection operators.
- single state,
- composite state,
- *pure state*, when the density matrix corresponds to a single state, it is a projection operator that projects onto that state. Furthermore: $\rho^2 = \rho$, and $Tr(\rho^2) = 1$.
- entangles state,
- *maximum entangled state*, $|\text{correlation}| = 1$.
- *partial entangled state*, $0 < |\text{correlation}| < 1$.
- composite system,
- full composite system,
- subsystem,
- *mixed or entangled state,* $\rho^2 \neq \rho$, and $Tr(\rho^2) < 1$.

Some definitions are needed or clarifications?

Lecture 8. Particles and Waves.

This Lecture is about: "… *the nonclassical logical principles that govern their behaviour* [of

particles and waves]".
Linear operators and the state of a particle are discussed.

8.1 Mathematical Interlude: Working with Continuous Functions

8.1.1 Wave Functions Review

First the results of Lecture 5 are reviewed.
To remember, at the top of page 237: *"But notice: the specific form of $\psi(\lambda)$ depends on the specific observable $\boldsymbol{L}$ that we initially choose"*.
Remark:
In Lecture 8.1 Susskind writes, bottom page 237: *" You can think of the wave function in two ways. First, it is the set of components of the state vector in a basis. These components can be stacked up to form a column vector"*. See for example Lecture 1.9.3. These components are called the probability amplitudes or wave functions. Susskind continues: *"Another way to think of the wave function is a function of λ. If you specify any allowable value of λ, the (wave)function $\psi(\lambda)$ produces a complex number. …….When thought of in this way, linear operators become operations that are applied to functions, and give back new functions"*. I would have preferred a rigorous proof. We come back to that later.
Reminder, classical and non-classical:
$P(\lambda) = \psi^*(\lambda)\psi(\lambda)$.

8.1.2 Functions as Vectors

In this Lecture, Susskind explained why *"we have to expand the idea of vectors to include functions"*.
At the bottom of page 239 and at the top of page 240, Susskind summarized the axioms that define a complex vector space. Complex functions satisfy all of them (See Lecture 1.9.1).
To replace sums by integrals is to find a particle in a small segment dx.
Eq.(8.2) represents the continuous representation of the inner product.
Probability densities replace probabilities.
Dirac delta functions replace Kronecker delta.

The Dirac delta function is introduced, pages 242-245, and defined by Eq.(8.4).

In Chapter III on Representation, Dirac explained the need for the δ function to deal with certain infinities. Feynman Vol III 16-4 used the δ in the section on normalization of the states x. Furthermore Feynman is additional instructive reading about the position operator. In Chapter 20 Operators of Vol III Feynman presented an example of integration by parts in the section on The momentum operator.

8.1.3 Integration by Parts

This is a strong tool to evaluate integrals since *the wave function must go to zero at infinity*.

8.1.4 Linear Operators

Susskind rehearsed the concept of operators.
As an example the operation: multiply by x represented by the operator $\boldsymbol{X}$ and the operation d/dx represented by the operator $\boldsymbol{D}$.

Exercise 8.1 Prove the position operator and the differentiation operator to be linear.

Prove that $\boldsymbol{X}$ and $\boldsymbol{D}$ are linear operators.

- Operator acts on a function giving another function.
- Operator acts on the sum of two functions giving the sum of the individual functions.
- Operator acts on a complex numerical multiple of a function giving the same multiple of the original results.

See page 53:
$\boldsymbol{M}|A\rangle = |B\rangle$,
$\boldsymbol{M}z|A\rangle = z|B\rangle$ and
$\boldsymbol{M}(|A\rangle + |B\rangle) = \boldsymbol{M}|A\rangle + \boldsymbol{M}|B\rangle$.
As an example for a function we look at Eq.(8.5):
$X\psi(x) = x\psi(x)$. Assume $\psi(x)$ to be: $\psi(x) = f(x) + g(x)$, then
$\boldsymbol{X}\psi(x) = \boldsymbol{X}\big(f(x) + g(x)\big) = x\big(f(x) + g(x)\big) = xf(x) + xg(x)$.

Remark:
In this Lecture, Susskind showed the operator $\boldsymbol{D}$ not to be Hermitian. Instead of $\boldsymbol{D}$, he introduced $-i\hbar\boldsymbol{D}$. Plugging this operator in the Eqs.(8.7) and (8.9) , we find,

$$\langle\Psi|-i\hbar\boldsymbol{D}|\Phi\rangle = -i\hbar\int\psi^*\frac{d\phi(x)}{dx}dx,$$

and

$$\langle\Phi|-i\hbar\boldsymbol{D}|\Psi\rangle = -i\hbar\int\phi^*\frac{d\psi(x)}{dx}dx.$$

Integration by parts:

$$\langle\Phi|-i\hbar\boldsymbol{D}|\Psi\rangle = i\hbar\int\psi(x)\frac{d\phi^*(x)}{dx}dx = \langle\Psi|-i\hbar\boldsymbol{D}|\Phi\rangle^*$$

Showing $-i\hbar\boldsymbol{D}$ to be Hermitian.

8.2 The State of a Particle

First the state in classical mechanics is explained. Based on classical mechanics, a state in quantum mechanics is proposed. Susskind: *This is incorrect*, page 251. It is not position and momentum. However, it is position or momentum: *The two do not commute*. A result of experimental observations.

8.2.1 The Eigenvalues and Eigenvectors of Position

Susskind starts with the operator $\boldsymbol{X}$, the observable of position, and looks for the eigenvectors and eigenvalues of the eigen-equation:
$\boldsymbol{X}|\Psi\rangle = x_0|\Psi\rangle$,
where x_0 is the eigenvalue.
Then: *"In terms of wave functions, this becomes:*
$x\psi(x) = x_0\psi(x)$, *Eq.(8.11)".*
I refer to the above remark on wave functions. We learned that for a basis $|x\rangle$ the wave function (or probability amplitude) is found from the inner products of the state-vector onto the eigenvectors (page 237) . Can we apply this to the above eigen-equation (8.11)?
Well, let's have a look: x is a real number and $\boldsymbol{X}$ does not operate on x. This does not apply to the operator $\boldsymbol{D}$! So let's multiply $\langle x|$ into the eigen-equation: $\boldsymbol{X}|\Psi\rangle = x_0|\Psi\rangle$, we have
$\langle x|\boldsymbol{X}|\Psi\rangle = \langle x|x_0|\Psi\rangle$.
Then we find $\boldsymbol{X}\langle x|\Psi\rangle = x_0\langle x|\Psi\rangle$. With the definition of the wave function, $\langle x|\Psi\rangle = \psi(x)$,
$\boldsymbol{X}\langle x|\Psi\rangle = \boldsymbol{X}\psi(x) = x_0\psi(x)$.

Eq. (8.5),
$\boldsymbol{X}\psi(x) = x\psi(x)$,
changes the eigen-equation into
$x\psi(x) = x_0\psi(x)$, Eq.(8.11).
I feel a bit more comfortable with the above approach.
Remember in the discrete situation we would have written for $\psi(x)$: ψ_x.
On page 253, the workings of the Dirac delta function is shown, leading to the conclusion:
$\psi(x) = \delta(x - x_0)$.
Note:On page 254, in the middle, dx is missing in the integral:$\int_{-\infty}^{\infty} \delta(x - x_0)\psi(x)dx$. A typo.
Susskind writes: *"By the definition of delta functions given in Eq.(8.4), this integral evaluates to* $\langle x_0|\Psi\rangle = \psi(x_0)$". I prefer to start with $\langle x_0|\Psi\rangle = \psi(x_0)$ and follow Feynman. We have by the definition of $\delta(x)$:
$\psi(0) = \int \delta(x)\psi(x)dx$**.** Now change the argument of the delta function from x to $x - x_0$ and we obtain: $\int \delta(x - x_0)\psi(x)dx = \psi(x_0)$.
On the other hand, Susskind derived Eq. (8.13),
$\langle x|\Psi\rangle = \psi(x)$,
by using the Dirac delta function. In this way, he proved the assumption $x\psi(x) = x_0\psi(x)$ to represent the eigen-equation $\boldsymbol{X}|\Psi\rangle = x_0|\Psi\rangle$.
Now something different. The position eigenfunction with eigenvalue x_0 is:
$\psi(x) = \delta(x - x_0)$.
What does this tell us about the expectation value of $\boldsymbol{X}$**?**
Well, we know: $\langle \boldsymbol{X}\rangle = \int \psi^*(x)x\psi(x)dx = \int \delta(x - x_0)x\delta(x - x_0)dx$.
We expect this expectation value to be x_0, the eigenvalue of the operator.
So $\int \delta(x - x_0)x\delta(x - x_0)dx = x_0$.
Furthermore, the normalization condition is: $\int \delta(x - x_0)\delta(x - x_0)dx = 1$.
This condition is given by Dirac, page 60, Eq. (10), by setting $b = a = x_0$ and make use of Eq. (6), on the same page, $\delta(-x) = \delta(x)$.
$\delta(x - x_0)\delta(x - x_0)$ has the same character as $\delta(x - x_0)$:
$\delta(x - x_0)\delta(x - x_0) = 0$ for $x \neq x_0$,
and becomes sufficient large for $x = x_0$ and,
so, $\int \delta(x - x_0)\delta(x - x_0)dx = 1$.
The least we can say: $\delta(x - x_0)\delta(x - x_0) \propto \delta(x - x_0)$,
since we have $\int \delta(x - x_0)\delta(x - x_0)dx = 1 = \int \delta(x - x_0)dx$.
Hence:
$(\delta(x - x_0))^2 \equiv \delta(x - x_0)$. With " ≡ " I mean identical behaviour.
Dirac did not mention this. I assume he considered this to be trivial.
In addition we can write $(\delta(x - x_0))^3 = \delta(x - x_0)(\delta(x - x_0))^2 \equiv \delta(x - x_0)\delta(x - x_0) \equiv \delta(x - x_0)$. With proof by induction, we find for $n \in \mathbb{N}$: $(\delta(x - x_0))^n \equiv \delta(x - x_0)$.
Basically I consider the above derivation not to be elegant. More or less we imply $\langle \boldsymbol{X}\rangle = x_0$.
Can we do better?
Let's replace x by $y + x_0$:
$\langle \boldsymbol{X}\rangle = \int_{-\infty}^{\infty} \delta(y)(y + x_0)\,\delta(y)dy = \int_{-\infty}^{\infty} \delta(y)y\delta(y)dy + x_0 \int_{-\infty}^{\infty} \delta(y)\delta(y)dy$.

For the first integral on the right hand side we use Dirac's toolkit, page 60- Eq. (7):
$y\delta(y) = 0$.
The second integral on the right-hand side gives with the normalization condition: x_0.
Consequently: $\langle \boldsymbol{X} \rangle = x_0$.
More concise : $\langle \boldsymbol{X} \rangle = \langle y + x_0 \rangle = \langle y \rangle + x_0 = x_0$.

8.2.2 Momentum and Its Eigenvectors

The momentum operator $\boldsymbol{P}$ is presented in Eq.(8.14), the relation between the momentum operator $\boldsymbol{P}$ and the differentiation operator $\boldsymbol{D}$.
The eigen-equation, in vector notation, Eq.(8.16):
$\boldsymbol{P}|\Psi\rangle = p|\Psi\rangle$ or
using the wave function:
$-i\hbar \frac{d\psi(x)}{dx} = p\psi(x)$ or
$\frac{d\psi_p(x)}{dx} = \frac{ip}{\hbar}\psi_p(x)$. **(L.8.1)**
The subscript p is a reminder that $\psi_p(x)$ is a wave function of $\boldsymbol{P}$ with the specific eigenvalue p in the position representation. At the bottom of page 256 Susskind denoted $\psi_p(x)$ to be the eigenvector of $\boldsymbol{P}$. I consider that to be a bit confusing, it is about the wave function.
Finally, the expression for $\psi_p(x)$ is obtained by integrating **(L.8.1)**, page 256,
$\psi_p(x) = Ae^{\frac{ipx}{\hbar}}$. **(L.8.2)**
The factor A is found by normalizing **(L.8.2)**. Susskind: "*Normalizing the eigenvectors is a more subtle operation, but the result is simple*".
$A = 1/\sqrt{2\pi}$.
Subtle indeed.
In the citation of Feynman, I used the Susskind notation:
Feynman: "There are several ways the normalization can be adjusted. We will choose one of them which we think to be the most convenient, although that may not be apparent to you just now" , Vol III, 16-3.
Then, Feynman looked at an example- for instance one in which a particles localized in a certain region around $x = 0$. Feynman choses a sort of smeared out delta function and obtained an expression for $\psi_p(x)$. You can cheque the normalization condition by plugging this expression into the probability distribution $|\widetilde{\psi_p}(p)|^2 \frac{dp}{2\pi\hbar}$ and integrate. You will find this to be equal to 1. Feynman: *"With the normalization chosen for the probability distribution the proper constant A (Susskind notation) is just 1".*
Maybe we need a dialogue Feynman-Susskind, one like the dialogue Democritus- Lederman (*The God Particle*)? Of some help could be the integral representation of the δ-function:
$\delta(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iky}dk$, (Chisholm and Morris*)*.
Well, let's find out whether this helps us to find out about A (Noordzij, 2).
We define two eigenvalues p and p'.
Now with the integral representation of the delta function, $k = x$ and $y = (p - p')/\hbar$:
$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix(p-p')/\hbar}dx = \delta[(p - p')/\hbar]$.
Well, you see it coming:

$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix(p-p')/\hbar}dx = \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-ix(p')}\frac{1}{\sqrt{2\pi}}e^{ix(p)/\hbar}dx = \delta[(p-p')/\hbar]$.

The integral represents the product of a plane wave eigenfunction with eigenvalue p and a complex conjugate wave eigenfunction with eigenvalue p'. This expression shows the compliance with the normalization and orthogonality condition. The delta function on the right-hand side can be written as, Dirac(page 60), $\hbar\,\delta(p-p')$.

It is a kind of magic, the Dirac δ-function. To have a proper delta function on the right-hand side, take note we find in this way A to be $1/\sqrt{2\pi\hbar}$. See the exercise below on normalization with Dirac Delta Function Approximation and the "Feynman factor" $\frac{1}{2\pi\hbar}$. Fitzpatrick also obtained the factor $1/\sqrt{2\pi\hbar}$. What about the "dimensionality" of the wave function? I leave that question.

Mahan indeed shows how to obtain the above constant A with help of delta-function normalization. However, using the plane wave function. There is no need for that.

We will do an additional exercise on normalization, next page.

Exercise on normalization with Dirac Delta Function Approximation

We will use the approximation of Susskind $\delta(x) = \frac{n}{\sqrt{\pi}} e^{-(nx)^2}$, pages 243-244, Fig.8.1.

So,

$\psi(x) = \frac{n}{\sqrt{\pi}} e^{-(nx)^2}$,

an approximation with the particle localized in a certain region around $x = 0$.

However, this wave function is not normalized. We must plug in a constant, B say, to be flexible. Then,

$\psi(x) = B \frac{n}{\sqrt{\pi}} e^{-(nx)^2}$.

With the normalization condition $\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1$,

we find for B: $B = (\frac{\sqrt{2\pi}}{n})^{1/2}$ and $\psi(x) = (n\sqrt{\frac{2}{\pi}})^{1/2} e^{-(nx)^2}$.

In this case we can still choose a number for n. For $B = 1$, the normalization condition gives: $n = \sqrt{2\pi}$. The number n is fixed.

Figure 8.1 gives an idea what this approximation looks like.

Now we have:

$\psi_p(x) = Ae^{\frac{ipx}{\hbar}}$, and we want to find A by normalization. We start with the wave function in the momentum representation (page 263):

$\tilde{\psi}(p) = \int dx \langle p|x\rangle \langle x|\Psi\rangle$.

$\langle p|x\rangle = \psi_p(x) = Ae^{\frac{ipx}{\hbar}}$ and we use for $\langle x|\Psi\rangle$ the delta function approximation, the result for $\tilde{\psi}(p)$ is :

$\tilde{\psi}(p) = A(n\sqrt{\frac{2}{\pi}})^{1/2} \int_{-\infty}^{\infty} \exp(-\frac{ipx}{\hbar} - (nx)^2)dx$.

This can be written as:

$A(\frac{1}{n}\sqrt{\frac{2}{\pi}})^{1/2} \exp(-(\frac{p}{2n\hbar})^2) \int_{-\infty}^{\infty} \exp(-(nu)^2)dnu = A(\frac{1}{n}\sqrt{\frac{2}{\pi}})^{1/2} \exp\left(-\left(\frac{p}{2n\hbar}\right)^2\right)\sqrt{\pi} = \tilde{\psi}(p)$.

(Feynman citation: "The mathematicians would probably object to the way we got there, but the result is nevertheless correct".)

Feynman used for the Dirac Delta Function Approximation: $\psi(x) = K \cdot \exp\left(-\left(\frac{x}{2\sigma}\right)^2\right)$. σ is a measure for the half-width of the curve and K a constant to be determined by normalization. The remark by Feynman about mathematicians has something to do with integrating a complex function.

Now $\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\tilde{\psi}(p)dp = 1$.

Then $A^2 \frac{\sqrt{2\pi}}{n} \int_{-\infty}^{\infty} \exp(-2(\frac{p}{2n\hbar})^2)dp = 1$.

This leads to:

$A = (\frac{1}{2\hbar\sqrt{\pi}})^{1/2}$. Well, this is not near $\frac{1}{\sqrt{2\pi}}$.

What about the dimension of A? This cannot be correct. Feynman, to find A, used for the probability distribution $|\tilde{\psi}(p)|^2 \frac{dp}{\hbar}$. Then, with the normalisation condition $\int_{-\infty}^{\infty} \tilde{\psi}^*(p)\tilde{\psi}(p)\frac{dp}{2\pi\hbar} = 1$, $A = 1$.

However, Susskind found $A = \frac{1}{\sqrt{2\pi}}$. When we take for the probability distribution $|\tilde{\psi}(p)|^2 \frac{dp}{\hbar}$, we finally obtain $\frac{1}{\sqrt{2\pi}}$. *The Feynman factor* $\frac{1}{2\pi\hbar}$ *is found by trial and error?* Or is it the magic of the genius? For sure, I found the factor $\frac{1}{\hbar}$ by trial and error.

Again: subtle.

On top of page 258 Susskind writes: *"……Eq. (8.18). The second equation is simply the complex conjugate of the first. These results are easy to verify if you keep in mind that $|x\rangle$ is represented by a delta function"*. Verify what? That the two functions are the complex conjugate of each?

$\langle x|p\rangle = \langle p|x\rangle^*$.

Or we need to verify $\psi_p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{ipx}{\hbar}}$?

Well, in general $\langle \Phi|\Psi\rangle = \int_{-\infty}^{\infty}\langle \Phi|x\rangle\langle x|\Psi\rangle dx$.
Then,

$\langle x_0|p\rangle = \int_{-\infty}^{\infty}\delta(x-x_0)\,\psi_p(x)dx = \psi_p(x_0) = \frac{1}{\sqrt{2\pi}}e^{\frac{ipx_0}{\hbar}}$.

Since this applies for any x_0: $\psi_p(x) = \langle x|p\rangle = \frac{1}{\sqrt{2\pi}}e^{\frac{ipx}{\hbar}}$.

Furthermore on page 258, point 2. Susskind writes: *"We have been using the symbol ψ for both position and momentum eigenstates."* I still think the symbol for eigenstates is Ψ.
On the pages 258-260, Susskind reflects on the wave function, momentum, and wavelength.

8.3 Fourier Transforms and the Momentum Basis

The importance of the wave function for determining the probability of finding a particle at position x is mentioned on page 260. Again, this presentation was proposed by Born.
So, completely similar, the probability of a particle having momentum p is obtained.
This is presented in Eq. (8.19).
I prefer the notation: $P(p) = |\langle p|\Psi\rangle|^2$.
And Eq.(8.20): $\tilde{\psi}(p) = \langle p|\Psi\rangle$. See top of page 263.
As mentioned by Susskind, there is a relation between the momentum representation and the position representation. Both represent the state-vector: $|\Psi\rangle$.

8.3.1 Resolving the Identity.

The trick is resolving the identity using the identity operator $\boldsymbol{I}$, derived for the discrete case, for the continuous case.
In Eqs. (8.21) and (8.22) , the identity operator is presented twice: in the two basis vectors.
On page 263, Susskind elegantly presented the transformation of a wave function in the position representation into the wave function in the momentum representation.
On page 264 their relationship with Fourier analysis is mentioned.

8.4 Commutators and Poisson Brackets.

Two important principles about commutators:
- The connection between classical and quantum mechanics.
- Uncertainty.

Susskind discussed the connection between quantum mechanics and classical mechanics. It is about commutators and Poisson brackets to some extent represented by
Eq. (8.26), based on Eq.(4.21),
$[\boldsymbol{L},\boldsymbol{M}] \leftrightarrow i\hbar\{L,M\}$. $\boldsymbol{L}$ and $\boldsymbol{M}$ are operator symbols.
Susskind: *"….we're reminded that the equations for quantum motion strongly resemble their classical equivalents."*
In Eq.(8.29) the commutator $[\boldsymbol{X},\boldsymbol{P}]$ is derived from Eqs. (8.27) and (8.28).

In Chapter IV On Quantum Conditions, Dirac gave the expression $uv - vu = i\hbar\{u,v\}$, Eq. 7. Here we used the Susskind notation for Poisson Brackets. Furthermore in Eq. 7, u and v are dynamical variables. As defined by Dirac, dynamical variables whose eigenstates are a complete set are observables. For the canonical coordinate x and canonical momentum p, Eq. 7 of Dirac can be written with the notation of Susskind as $[x,p] = i\hbar\{x,p\}$.

Page 268, $\{x, p\} = 1$ and consequently $[x, p] = i\hbar$. Again, square brackets represent a commutator.

8.5 The Heisenberg Uncertainty Principle

See Lecture 5 to read about the general uncertainty principle.
Eq.(5.13), the simultaneous uncertainties of the observables $\boldsymbol{A}$ and $\boldsymbol{B}$

$\Delta\boldsymbol{A}\Delta\boldsymbol{B} \geq \frac{1}{2}||\langle\Psi|[\boldsymbol{A}, \boldsymbol{B}]|\Psi\rangle$.

The position and momentum operators are plugged into the preceding expression, bottom page 269.
Susskind writes, page 270: "*On the other hand, the probability $P(x)$ for a momentum eigenstate is uniformly spread over the x axis. To see this, let's take the wave function, Eq. (8.17),*

$\psi_p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{ipx}{\hbar}}$ *and multiply it by its complex conjugate:*

$\psi_p^*(x)\psi_p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{ipx}{\hbar}} \frac{1}{\sqrt{2\pi}} e^{\frac{ipx}{\hbar}} = \frac{1}{2\pi}$*. The result is completely uniform*".
Now we also know, Eq.(8.3):

$\int P(x)dx = \int \psi_p^*(x)\psi_p(x)dx = \int_{-\infty}^{\infty} \frac{1}{2\pi} dx = 1.$

Is this a surprising result? Well, the result is wrong. Here the Heisenberg uncertainty principle is at work:
$\int P(x)dx \to \infty$, since as mentioned by Susskind, the eigenstate of the position is a Delta function giving the position exactly at x_0. Consequently $\Delta\boldsymbol{X} \to 0$ and $\Delta\boldsymbol{P} \to \infty$.
Well, there is still something uncomfortable here.

Note: The probability $P(x)$ is uniformly spread over the x-axis. Let's start with a uniform distribution over an interval $(0, a)$. $P(x) = A$, a constant over the given interval. Then normalization gives:

$\int_0^a P(x)dx = Aa = 1 \to A = \frac{1}{a}$.

Hence, for a uniform distribution spread over the whole x-axis: $a \to \infty$ and $A \to 0$

Question:
The wave equation. Susskind mentioned on page 258, $\psi(x)$ to be *"…just the generic symbol for whatever function we happen to be discussing"*. Now we used in Lecture 8 various wave functions:

- $\psi(x) = \delta(x - x_0)$: this wave function represents states in which the particle is located right at the point x_0 on the x axis; the position representation, page 253.
- $\psi_p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{ipx}{\hbar}}$: the momentum eigenfunction of the operator $\boldsymbol{P}$ in the position basis, Eq.(8.17). The basis vectors are: $|x\rangle$. A wave function I suppose? Feynman: *if a particle has a definite momentum p and a corresponding definite energy E, the amplitude(wave function) to be found at any position x would look like* $\langle x|\Psi\rangle \propto e^{\frac{ipx}{\hbar}}$. So $\psi(x)$ and $\psi_p(x)$ should not be muddled together.

- $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{ipx}{\hbar}} \psi(x)$: the wave function in the momentum representation, Eq. (8.24). No dependence on x. The basis vectors are : $|p\rangle$.

We substitute $\psi(x) = \delta(x - x_0)$ into $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{ipx}{\hbar}} \psi(x)$ and obtain

$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} \int dx\, e^{-\frac{ipx}{\hbar}}\, \delta(x - x_0) = \frac{1}{\sqrt{2\pi}} e^{-\frac{ipx_0}{\hbar}} \neq \frac{1}{\sqrt{2\pi}} e^{\frac{ipx}{\hbar}} = \psi_p(x).$

However, one could imagine $\frac{1}{\sqrt{2\pi}} e^{-\frac{ipx_0}{\hbar}}$ to be valid for any x_0. So, $\neq \rightarrow =$.

Let us have a look at Eq. (8.25): $\psi(x) = \frac{1}{\sqrt{2\pi}} \int dp\, e^{\frac{ipx}{\hbar}} \tilde{\psi}(p)$.

Substitute the above expression $\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi}} e^{-\frac{ipx_0}{\hbar}}$ into the Eq. 8.25, with the result:

$\psi(x) = \frac{1}{2\pi} \int dp\, e^{\frac{ip(x-x_0)}{\hbar}}$.

Then, with $\psi(x) = \delta(x - x_0)$, the Dirac delta function:

$\delta(x - x_0) = \frac{1}{2\pi} \int dp\, e^{\frac{ip(x-x_0)}{\hbar}}$.

This expression for the Dirac delta function has the same form as the representation of the δ-function given above. However, to be the same, $\delta(x - x_0)$ should have been

$\delta(x - x_0) = \frac{1}{2\pi} \int \frac{dp}{\hbar} e^{\frac{ip(x-x_0)}{\hbar}}$,

A factor ħ is missing in front of the integral $\delta(x - x_0)$, since,

$\delta(x - x_0) = \frac{\hbar}{2\pi} \int \frac{dp}{\hbar} e^{\frac{ip(x-x_0)}{\hbar}}$.

If we scale x in : $\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dk$ (Chisholm and Morris) with ħ, then,

$\delta(x) = \frac{\hbar}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{\hbar} e^{\frac{ikx}{\hbar}}$ and this δ-function equals $\delta(x - x_0)$.

In Chisholm and Morris, the integral representation of the δ-function has been derived not with help of quantum mechanics. The integral representation is derived in a pure mathematical way. So, this δ-function representation can be used to find the above-mentioned factor A, page 257. Not knowing: $A = \frac{1}{\sqrt{2\pi}}$, we have

- $\psi(x) = \delta(x - x_0)$.
- $\psi_p(x) = Ae^{\frac{ipx}{\hbar}}$.
- $\tilde{\psi}(p) = A \int dx e^{-\frac{ipx}{\hbar}} \psi(x)$.

We substitute $\psi(x) = \delta(x - x_0)$ into $\tilde{\psi}(p) = A \int dx e^{-\frac{ipx}{\hbar}} \psi(x)$ and obtain

$\tilde{\psi}(p) = A \int dx\, e^{-\frac{ipx}{\hbar}}\, \delta(x - x_0) = Ae^{-\frac{ipx_0}{\hbar}}$.

Furthermore, we have:

$\psi(x) = \delta(x - x_0) = A \int dp\, e^{\frac{ipx}{\hbar}} \tilde{\psi}(p)$.

With $\tilde{\psi}(p) = Ae^{-\frac{ipx_0}{\hbar}}$,

$\delta(x - x_0) = A^2 \int dp\, e^{\frac{ip(x-x_0)}{\hbar}}$.

Use can be made of the integral representation for the δ-function as derived by Chisholm

and Morris with the result $A^2 = \frac{1}{2\pi}$. A subtle operation? I do not know.
In this Lecture Susskind summarizes the Heisenberg Uncertainty Principle elegantly. He showed that the probability $P(x)$ for a momentum eigenstate is uniformly spread over the x axis: $P(x) = \psi_p^*(x)\psi_p(x)$, and $\int P(x)dx = \int \psi_p^*(x)\psi_p(x)dx = \int_{-\infty}^{\infty} \frac{1}{2\pi} dx = 1$. Is it?
Eq.(8.25): $\psi(x) = \frac{1}{\sqrt{2\pi}} \int dp\, e^{\frac{ipx}{\hbar}} \tilde{\psi}(p)$.
We know $\psi_p(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{ipx}{\hbar}}$. Substitute this expression into Eq. 8.25 and we find $\psi(x) = \int dp\, \tilde{\psi}(p)\psi_p(x)$; $\psi(x) = \delta(x - x_0)$.
So finally:
$\delta(x - x_0) = \int dp\, \tilde{\psi}(p)\psi_p(x)$. Is this to be expected?

Lecture 9. Particle Dynamics.

9.1 A Simple Example

To start with: *the minus first law*: States change in a way that information and distinctions are never erased. This leads to the principle of unitary.
Now Susskind paid attention to: *How do particles move in quantum mechanics?*
Then Susskind introduced the *Original Schrödinger Equation*, a special case of Eq.(9.1), representing the time-dependent Schrödinger equation.
A simple version of the Hamiltonian is analysed, Eq.(9.2)
With this simple Hamiltonian, the expectation value of position behaves according to classical equations of motion.

Remark:
On page 276 Susskind derived the derivatives of a function depending on $(x - ct)$. I feel a bit more comfortable by writing $\frac{\partial}{dx}\psi(x - ct) = \frac{\partial\psi}{\partial(x-ct)}\frac{\partial(x-ct)}{\partial x} = \frac{\partial\psi}{\partial(x-ct)}$,
and
$\frac{\partial}{\partial t}\psi(x - ct) = \frac{\partial\psi}{\partial(x-ct)}\frac{\partial(x-ct)}{\partial t} = -c\frac{\partial\psi}{\partial(x-ct)}$.
Then,
$\frac{\partial}{\partial t}\psi(x - ct) = -c\frac{\partial}{dx}\psi(x - ct)$.
Hence, for this whole family of solutions for ψ , depending on $x - ct$ we have the operator $\frac{\partial}{\partial t} = -c\frac{\partial}{dx}$, [Susskind, Volume III, page 131, there the function $\psi(x + ct)$ is analysed].
Notice that the ∂ is missing on page 276 in the derivative $\frac{\partial}{\partial x}$. A typo.

9.2 Non-relativistic Free Particles.

Remember: *"Only massless particles can move at the velocity of light,…."*.
Gravitons?
An important remark, page 280: *"So, if you want to write down the quantum mechanical equations of a system whose classical physics you already know, it's very reasonable to try using the classical Hamiltonian, translated into operator form"*.
Here I also refer to Lectures 4.5 and 4.6, where the time dependent Schrödinger equation

has been derived, Eq.(4.10)

$\hbar \frac{\partial|\Psi\rangle}{\partial t} = -i\boldsymbol{H}|\Psi\rangle$.

Starting with the classical formulation of the Hamiltonian, Pages 280 and 281, Susskind presented the quantum mechanical equivalent in Eq.(9.4) for a nonrelativistic free particle. Equal to the result presented in Eq.(4.10) and

$\boldsymbol{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$.

9.3 Time-Independent Schrödinger Equation

In this Lecture the time-dependent Schrödinger-equation is solved. First the time-independent Schrödinger-equation is solved.

On page 284 Susskind used the expression "*eigenvectors*". I think it should be eigenfunctions. We get used to it.

Exercise 9.1 The kinetic energy

Derive Eq. 9.7 by plugging Eq. 9.6 into Eq. 9.5.

Substitute $\psi(x) = e^{\frac{ipx}{\hbar}}$, into

$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = E\psi(x)$.

$\frac{\partial^2\psi(x)}{\partial x^2} = \left(\frac{ip}{\hbar}\right)^2 \psi(x)$.

So,

$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} = \frac{p^2}{2m}\psi(x) = E\psi(x)$.

Consequently,

$E = \frac{p^2}{2m}$.

Nota bene: in this case $\psi(x) = \psi_p(x)$ (See page 256) and Susskind writes (below Eq. (9.5)): "*..momentum eigenvectors do the job…*"

Remark:

Instead of $\psi(x) = e^{\frac{ipx}{\hbar}}$, we could also have substituted $\psi(x) = \tilde{\psi}(p)e^{\frac{ipx}{\hbar}}$, since $\tilde{\psi}(p)$ does not explicitly depends on x.

On page 285 the time-dependent wave function is obtained.

Remark and (a lot of) Questions:

On page 285 Susskind derived(constructed) the time-dependent wave equation with help of the time-independent Schrödinger equation. The momentum representation of the wave function:

$\psi(p,t) = \tilde{\psi}(p)\frac{1}{\sqrt{2\pi}}\exp\left(\frac{i(px-\frac{p^2t}{2m})}{\hbar}\right)$.,

where the factor $\tilde{\psi}(p)\frac{1}{\sqrt{2\pi}}$, has been plugged in.

Instead of $\frac{1}{\sqrt{2\pi}}$ we could plug into the expression for $\psi_p(x,t)$, $\tilde{\psi}(p)$ and obtain:

$\psi_p(x,t) = \tilde{\psi}(p)\exp\left(\frac{i(px-\frac{p^2t}{2m})}{\hbar}\right)$. (See Remark above, after Exercise 9.1)

Then Susskind writes: "*Any solution is a sum, or integral, of* $\psi(x,t) = exp\,(\frac{i\left(px-\frac{p^2t}{2m}\right)}{\hbar})$*:*

$\psi_p(x,t) = \int \tilde{\psi}(p) exp\,(\frac{i(px-\frac{p^2t}{2m})}{\hbar})dp$ ".

Well, intuitively I can understand this. However, I appreciate the following approach. With Eq.(8.25) we have:

$\psi_p(x) = \frac{1}{\sqrt{2\pi}}\int \tilde{\psi}(p)dp exp(\frac{ipx}{\hbar})$.

$\psi_p(x)$ is the time-independent wave equation in the coordinate presentation.

The time-dependent solution is

$\psi_p(x,t) = \psi_p(x)\exp\,(-\frac{iEt}{\hbar})$, or with Eq.(9.7), $E = p^2/2m$:

$\psi_p(x,t) = \frac{1}{\sqrt{2\pi}}\int \tilde{\psi}(p)\exp\,(\frac{i(px-\frac{p^2t}{2m})}{\hbar})dp$.

Still, I am not sure. Why? Well, using Eq.(8.25) for $\psi(x)$ and plug $\exp\,(-\frac{iEt}{\hbar})$ into the integral we find the expression for $\psi(t,x)$ or $\psi_p(t,x)$. Is this correct? The expression $\exp\,(-\frac{iEt}{\hbar})$ depends on p through $E = \frac{p^2}{2m}$ and the integral is over p.

Above I derived the expression $\psi_p(x,t) = \tilde{\psi}(p)\exp\,(\frac{i(px-\frac{p^2t}{2m})}{\hbar})$. To me it becomes a bit more clear that, with the latter expression, a sum over all p or an integral over p leads to

$\psi_p(x,t) = \int \tilde{\psi}(p)\exp\,(\frac{i(px-\frac{p^2t}{2m})}{\hbar})dp$.

On page 285 Susskind writes below the preceding expression: "*You can start with any wave function [*$\psi_p(x)$*?] at* $t = 0$*, find* $\tilde{\psi}(p)$ *by Fourier transformation, and let it evolve…….*". This simple general solution has an important implication. Among other things, it says that the wave function in momentum-representation ($\tilde{\psi}(p)$?) changes with time in a remarkably simple way:

$\tilde{\psi}(p,t) = \tilde{\psi}(p)\exp\,(\frac{i(px-\frac{p^2t}{2m})}{\hbar})$ ". **(L9.1)**

Can this be right? Susskind derived:

$\psi_p(x,t) = \int \tilde{\psi}(p)\exp\,(\frac{i(px-\frac{p^2t}{2m})}{\hbar})dp$.

Now we substitute the expression for $\tilde{\psi}(p,t)$ into the above expression for $\psi_p(x,t)$. We find:

$\psi_p(x,t) = \int \tilde{\psi}(p,t))dp$.

Well, this seems to be wrong. So? In Eq. **(L9.1)** $\tilde{\psi}(p,t)$ also depends on x. Consequently, the phase changes in addition to t, with x. Or should $\tilde{\psi}(p,t)$ in Eq. (**L9.1)** be written as $\tilde{\psi}(p,t) = \tilde{\psi}(p)\exp\,(-\frac{ip^2t}{2m})$? Just like $\psi_p(x,t) = \psi_p(x)\exp\,(-\frac{ip^2t}{2m})$?

Or, again with Eq.(8.25),

$\tilde{\psi}(p,t) = \int \psi(x)\exp\,(\frac{i(-px-\frac{p^2t}{2m})}{\hbar})dx$?

Here $\tilde{\psi}(p,t)$ does not depend on x, and $\psi(x)$ is the position(or coordinate) representation of the momentum wave function. Can we still denote $\psi(x)$ to be the wave functions representing states in which the particle is located right at the position x_0 on the x axis

(pages 253/254)? Then
$\psi(x) = \delta(x - x_0)$ and
$\tilde{\psi}(p,t) = \exp\left(\frac{i(-px_0 - \frac{p^2 t}{2m})}{\hbar}\right)$.
With Eq.(8.24) and $\psi(x) = \delta(x - x_0)$ we have
$\tilde{\psi}(p) = \exp\left(-\frac{ipx_0}{\hbar}\right)$.
So,
$\tilde{\psi}(p,t) = \tilde{\psi}(p)\exp\left(-\frac{ip^2 t}{2m}\right)$.
I cannot explain the differences away. I doubt **(L9.1)** to be correct. A lot of questions indeed.

9.4 Velocity and Momentum

On page 286, Susskind presented the quantum mechanical formulation of the velocity.
Susskind advised to review Lecture 4, section 4.9.
It is about Eq.(4.17):
$\frac{d}{dt}\langle \boldsymbol{L}\rangle = \frac{i}{\hbar}\langle[\boldsymbol{H},\boldsymbol{L}]\rangle$.
Then, with $\boldsymbol{H} = \frac{\boldsymbol{P}^2}{2m}$, and $\boldsymbol{L} = \boldsymbol{X}$ **,** Eq.(9.9) is obtained.

Exercise 9.2 A Commutator

Prove Eq. 9.10:
$[\boldsymbol{P^2},\boldsymbol{X}] = \boldsymbol{P}[\boldsymbol{P},\boldsymbol{X}] + [\boldsymbol{P},\boldsymbol{X}]\boldsymbol{P}$ by expanding each side and comparing the results.
$\boldsymbol{P}[\boldsymbol{P},\boldsymbol{X}] + [\boldsymbol{P},\boldsymbol{X}]\boldsymbol{P} = \boldsymbol{P^2X} - \boldsymbol{PXP} + \boldsymbol{PXP} - \boldsymbol{XP^2} = \boldsymbol{P^2X} - \boldsymbol{XP^2} = [\boldsymbol{P^2},\boldsymbol{X}]$ **.**

For the velocity v we have $v = \frac{d}{dt}\langle \boldsymbol{\Psi}|\boldsymbol{X}|\boldsymbol{\Psi}\rangle = \frac{d}{dt}\langle \boldsymbol{X}\rangle$.
Furthermore, we know, with Eq. 4.17:
$\frac{d}{dt}\langle \boldsymbol{X}\rangle = \frac{i}{\hbar}\langle[\boldsymbol{H},\boldsymbol{X}]\rangle$.
Page 282: $\boldsymbol{H} = \frac{\boldsymbol{P^2}}{2m}$.
Then,
$v = \frac{d}{dt}\langle \boldsymbol{X}\rangle = \frac{i}{2m\hbar}\langle[\boldsymbol{P^2},\boldsymbol{X}]\rangle$.
$\boldsymbol{P}[\boldsymbol{P},\boldsymbol{X}] + [\boldsymbol{P},\boldsymbol{X}]\boldsymbol{P} = [\boldsymbol{P^2},\boldsymbol{X}]$ **.**
We also know $[\boldsymbol{P},\boldsymbol{X}] = -i\hbar$:
with $[\boldsymbol{P^2},\boldsymbol{X}] = \boldsymbol{P}[\boldsymbol{P},\boldsymbol{X}] + [\boldsymbol{P},\boldsymbol{X}]\boldsymbol{P}$
$[\boldsymbol{P^2},\boldsymbol{X}] = -2i\hbar\boldsymbol{P}$.

For v we finally obtain: $v = \frac{d}{dt}\langle \boldsymbol{X}\rangle = \frac{i}{2m\hbar}\left\langle[\boldsymbol{P^2},\boldsymbol{X}]\right\rangle = \frac{\langle \boldsymbol{P}\rangle}{m}$,
or $\langle \boldsymbol{P}\rangle = mv$, Eq.(9.11).
On page 288 below Eq.(9.11),Susskind writes: "*The expectation value of x….*". Eq. (9.11) is about the expectation value of $\boldsymbol{P}$**.** Furthermore, Susskind writes: *"What Eq. (9.11) tells us is that the centre of the wave packet travels according to the classical rule $p = mv$."* So, is it about the expectation value of x? I think it is. The confusion stems from the expectation value of $\boldsymbol{P}$**.**

9.5 Quantization

Susskind started this Lecture summarizing the results of Lecture 9 so far, the process of quantization, briefly:

- start with a classical system,
- replace the classical phase space with a linear vector space,
- replace the x's and p's by their operators,
- with these replacements, the Hamiltonian becomes an operator.

Susskind concluded this section: "*Quantum theory is probably much more fundamental than classical theory, which generally should be understood as an approximation*".

9.6 Forces

Susskind introduced the subject matter with the classical formulation of a force and the related potential.

Remark:

On page 291 Susskind discusses the operator $\boldsymbol{V}$ and writes: "*When the operator $\boldsymbol{V}$ acts on any wave function $\psi(x)$, it multiplies the wave function by the function $V(x)$*". To make the text comply with the expression below this text, the text should read: When the operator $\boldsymbol{V}$ acts on the vector $|\Psi\rangle$ it multiplies the wave function $\psi(x)$ by the function $V(x)$.

On page 293 Susskind writes: "*But multiplying by x and multiplying by a function of x are operations that commute. In other words:* $[\boldsymbol{X}, \boldsymbol{V}(x)] = 0$". Let's have a look.

We start with Eq.(9.13): $\boldsymbol{H} = \frac{\boldsymbol{P^2}}{2m} + \boldsymbol{V}(x)$.

What is of interest is the commutator $[\boldsymbol{H}, \boldsymbol{X}]$**.** Substitute the expression for $\boldsymbol{H}$ into the commutator. The result is:

$[\boldsymbol{H}, \boldsymbol{X}] = \left[\frac{\boldsymbol{P^2}}{2m}, \boldsymbol{X}\right] + [\boldsymbol{V}, \boldsymbol{X}]$.

Now, $[\boldsymbol{V}, \boldsymbol{X}]$ or $[\boldsymbol{X}, \boldsymbol{V}(x)]$ is zero when both $\boldsymbol{X}$ and $\boldsymbol{V}(x)$ are Hermitian. They are. Susskind already proved $\boldsymbol{X}$ to be Hermitian. What about $\boldsymbol{V}$? The operator for the potential energy $\boldsymbol{V}$ can be written as a polynomial of $\boldsymbol{X}$. Consequently $[\boldsymbol{X}, \boldsymbol{V}(x)] = 0$. See page 115.

This can also be illustrated by Eq. (9.17) with $\boldsymbol{P}$ replaced by $\boldsymbol{X}$:

$[\boldsymbol{X}, \boldsymbol{V}(x)]\psi(x) = xV(x)\psi(x) - V(x)x\psi(x) = 0$.

Eq. (9.15) is obtained with, Eq.(4.17):

$\frac{d}{dt}\langle \boldsymbol{L} \rangle = \frac{i}{\hbar}\langle [\boldsymbol{H}, \boldsymbol{L}] \rangle$, and with Eq.(9.13) → Eq.(9.15) .

Exercise 9.3 Commutator of a potential function and the momentum operator

Show that the right-hand side of Eq. (9.17) simplifies to the right-hand sight of Eq.(9.16).

The right-hand side of Eq. (9.17):

$V(x)\left(-i\hbar\frac{d}{dx}\right)\psi(x) - \left(-i\hbar\frac{d}{dx}\right)V(x)\psi(x) =$

$-i\hbar V\frac{d\psi}{dx} + i\hbar V\frac{d\psi}{dx} + i\hbar\psi\frac{dV}{dx} = i\hbar\psi\frac{dV}{dx} = (i\hbar\frac{dV}{dx})\psi$.

Then , Eq.(9.16):

$[\boldsymbol{V}(x), \boldsymbol{P}]\psi = (i\hbar\frac{dV}{dx})\psi$

So, Eq.(9.16),

$[\boldsymbol{V}(x), \boldsymbol{P}] = i\hbar\frac{dV}{dx}$.

9.7 Linear Motion and the Classical Limit

Attention is paid to *the difference between the average of a function and the function of the average.*
Susskind emphasized, on page 295, *"… the classical equations are only approximations, …"*.
In the legenda of Figure 9.3, page 296, Susskind writes: "*Note* $\langle x \rangle = 0$, $\Delta x > 0$." As defined in Figure 8.2 ? I suppose so.
On the pages 296-301, Susskind discussed wave packets, potentials, and the uncertainty relation.

9.8 Path Integrals

Feynman's favourites. The principle of least action.
In this Lecture Susskind explained the least action principle starting with Volume I of the Theoretical Minimum Series.
Action: *the integral of the Lagrangian between the end points of the trajectory*, bottom page 302.
Next, the step is made into the quantum mechanical world. Keeping in mind the uncertainty principle, we deal with probabilities.
In Eq.(9.26), Susskind summarized Feynman's formulation.

Remark:
At the bottom of page 307 the tiny time interval should read "ϵ" instead of "e". Typo.
Time dependency of the state is presented with H in the exponential expression instead of E for the energy eigenvalue.

Lecture 10. Harmonic Oscillator.

This Lecture is about the classical and quantum mechanical description of the harmonic oscillator. *A basic ingredient of quantum mechanics.*
On the pages 312 and 313, Susskind presented a couple of examples of the harmonic oscillator.
Let us start by citing Dirac:" This *different algebra for the dynamical variables is one of the most important ways in which quantum mechanics differs from classical mechanics. We shall see later that, despite this fundamental difference, the dynamical variables of quantum mechanics still have many properties in common with their classical counterparts and it will be possible to build up a theory of them closely analogous to the classical theory and forming a beautiful generalization of it".*
Various examples of the harmonic oscillator can be found in Feynman Vol. I, Chapter 23.

10.1 The Classical Description

Susskind invited us *"… to imagine a very tiny version of a weight hanging from a spring".*
The Lagrangian of such a system is presented, Eq.(10.3). On page 315, Susskind use Lagrange's equation as presented in Volume I.

Exercise 10.1 The derivatives of a general oscillatory function

Find the second derivative of x in Eq.(10.9), and thereby show that it solves Eq.(10.8).
Eq.(10.9):
$x = Acos(\omega t) + Bsin(\omega t)$.
Then
$\ddot{x} = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t) = -\omega^2 x$.
This represents Eq.(10.8).

10.2 The Quantum Mechanical Description

In this Lecture, Susskind analysed the microscopic harmonic oscillator. A couple of examples are presented..

Remark:
Now that we are used to the conundrum of notations like $\psi(x), \psi_p(x), \psi(p), |\psi(x)\rangle$,..... $|\psi(p)\rangle$ and $|\Psi\rangle$, Susskind told us, page 317: "*There are many possible system states, and each one is represented by a different wave function"*.
In Lecture 8 we learned the wave function for a particle moving on a line to be represented by the Dirac Delta Function, page 253, $\psi(x) = \delta(x - x_0)$. So, the above statement about different wave functions is a general statement.
The Dirac Delta Function is just a mathematical construct to describe the position wave function of a point-like particle. Well, we know there is no such thing as a point-like particle. That is probably the reason why Susskind and Feynman used Dirac Delta Functions. Approximations: a bell type curve or smeared out Dirac Delta Function. See the above Exercise on normalization with Dirac Delta Functions Approximations in my notes on Lecture 8. Or is it the Heisenberg uncertainty principle that makes Feynman and Susskind choose a bell type curved wave function?

Page 318: It is about "… *a sensible wave function*…". Consequently, these functions need to be "*normalizable"*.
The Hamiltonian is also derived from the Lagrangian. Then, the Hamiltonian is derived in terms of operators, page 320.

10.3 The Schrödinger Equation

In this Lecture Susskind dealt with the time dependent Schrödinger Equation, Eq.(10.3), derived in Lecture 4.5, Eq. (4.9) and with the correct dimensions in Eq. (4.10):
$\hbar \frac{\partial |\Psi\rangle}{\partial t} = -i\boldsymbol{H}|\Psi\rangle$.
With Eq.(10.12), and the preceding equation, Eq.(10.13) is obtained.
Eq.(10.13) can be solved numerically.
" ….it will form a wave packet that moves around like a harmonic oscillator."
Feynman, et al, presented the Schrödinger equation for the motion of a particle along a line. See Vol. III Chapter 16, page 16-4. Attention is paid to the history of the equation.

10.4 Energy Levels

This Lecture is about calculating energy levels with the Hamiltonian.
A rehearsal of the Schrödinger Ket recipe of section 4.13 is advised.
On the top of page 323 two bullet points are given. The first mentions allowable values of E and the second possible eigenvalues of the energy. May be the difference between allowable and possible is subtle.
Examples of the one-dimensional solution of the Schrödinger Equation can be found in Chapter 2 of the book by Mahan: *"Quantum Mechanics in a Nutshell".*

On page 323 Susskind stated: *"Physical solutions of the Schrödinger equation must be normalizable."*
The issue here is: find those solutions.

10.5 The Ground State

The Lecture started with the question: *"What is the lowest possible energy level for a harmonic oscillator?"→"…and in fact, it has no state with zero energy either".*
The lowest energy level is a level to remember. It is called the ground state.
Susskind presented with Eq. (10.15) a wave function that works.
On page 326 and 327, Susskind derived the ground state energy, Eq.(10.16).

Remark about the ground state of the harmonic oscillator:
The ground state wave function:
$\psi_0(x) = \exp\left(-\frac{\omega}{2\hbar}x^2\right)$, Eq.(10.15).
This wave function is not normalized.
We could normalize $\psi_0(x)$ in the usual way.
$\int_{-\infty}^{+\infty} \psi_0^*(x)\psi_0 dx = 1$, Eq.(10.10).
With $\int_{-\infty}^{+\infty} e^{-y^2}dy = \sqrt{\pi}$, and $\psi_0(x) = A\exp\left(-\frac{\omega}{2\hbar}x^2\right)$,
we have $A = (\frac{\omega}{\hbar\pi})^{1/4}$. Caveat: this normalization is done in the x-space, not the real world.
Susskind used the transformation $x = \sqrt{m}y$, where y represents the real world. Experiments are done in the real world. So, what does $\psi(y)$ looks like? Using the transformation, we have:
$\psi(y) = A\exp(-\frac{\omega m}{2\hbar}y^2)$.
Normalization in the real world gives:
$A = (\frac{2}{\pi})^{1/4}(\frac{\omega m}{2\hbar})^{1/4}$.
On page 325 Susskind presented a guess for the ground state wave function of the harmonic oscillator, Eq.(10.15). And indeed, it is a guess. It is also called a trial function. The only thing you must do is to find out whether it works or not. Well, you can test this through substitution in the Schrödinger equation. How did the professor know? Well, I do not know. What we know is the potential of the harmonic oscillator to be a potential with the most attractive region near $x = 0$. Consequently, the particle is most probably found in the region near $x = 0$. So, the ground state wave function can be represented by a Gaussian

distribution as given in Fig. 10.1 on page 325.
On the other hand, since the approach is basically a guess or approximation, the guess can be tested by the so-called variational method to find the ground state (www.hitoshi.berkeley.edu) .
But now something different. In this case we could use a representation of the Dirac Delta Function (Chisholm and Morris): $\psi(x) = \delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\, dk$.
Then
$\frac{\partial^2\psi(x)}{\partial x^2} = -k^2\psi(x)$. The linear representation of the oscillator.
On page 326, Susskind showed the ground state wavefunction to be a good guess.

10.6 Creation and Annihilation operators

At the beginning of this Lecture, Susskind reflected on the similarity of the Heisenberg and the Schrödinger thinking about quantum mechanics. For the harmonic oscillator, the operator method is the more powerful tool. The commutator relations are involved.
As expected, Susskind applied the operator method to the harmonic oscillator. The commutation relation between the momentum operator and the position operator is particularly useful. The creation and annihilation operators are constructed with $\boldsymbol{P}$ and $\boldsymbol{X}$. At the bottom of page 328, the action of the creation(raising) and annihilation(lowering) operators are described.
On page 329, Eq.(10.18), Susskind starts working with the Hamiltonian for the harmonic oscillator expressed in the momentum and position operators:
$\boldsymbol{H} = \frac{1}{2}(\boldsymbol{P}^2 + \omega^2\boldsymbol{X}^2)$.
This expression for the Hamiltonian and at the bottom of page 330, it is shown how to recover the actual Hamiltonian by adding $\frac{\omega\hbar}{2}$. This additive factor can be ignored for now. On top of page 332 Susskind plugged this constant back into the Hamiltonian operator.
On the basis of factorizing the Hamiltonian, the definitions of the lowering, $\boldsymbol{a}^-$, and raising, $\boldsymbol{a}^+$, operators are presented in Eqs.(10.20) and (10.21).
In deriving Eq. (10.23), the Hamiltonian,
$\boldsymbol{H} = \omega\hbar\left(\boldsymbol{N} + \frac{1}{2}\right)$,
use has been made of $[\boldsymbol{X}, \boldsymbol{P}] = -[\boldsymbol{P}, \boldsymbol{X}]$ and $\boldsymbol{N} = \boldsymbol{a}^+\boldsymbol{a}^-$.
With Eqs.(10.20) and (10.21):
$[\boldsymbol{a}^+, \boldsymbol{a}^-] = \boldsymbol{a}^+\boldsymbol{a}^- - \boldsymbol{a}^-\boldsymbol{a}^+ = \frac{1}{2\omega\hbar}\{(\boldsymbol{P} + i\omega\boldsymbol{X})(\boldsymbol{P} - i\omega\boldsymbol{X}) - (\boldsymbol{P} - i\omega\boldsymbol{X})(\boldsymbol{P} + i\omega\boldsymbol{X})\} =$
$= \frac{1}{2\omega\hbar}\{2i\omega[\boldsymbol{P}, \boldsymbol{X}]\} = \frac{1}{2\omega\hbar}\{2i\omega(-i\hbar)\} = 1.$
Page 333 and page 334:
$[\boldsymbol{a}^+, \boldsymbol{N}] = \boldsymbol{a}^+\boldsymbol{N} - \boldsymbol{N}\boldsymbol{a}^+ = \boldsymbol{a}^+\boldsymbol{a}^+\boldsymbol{a}^- - \boldsymbol{a}^+\boldsymbol{a}^-\boldsymbol{a}^+ = \boldsymbol{a}^+(\boldsymbol{a}^+\boldsymbol{a}^- - \boldsymbol{a}^-\boldsymbol{a}^+)$.
The expression between brackets is the commutator of $\boldsymbol{a}^+$ and $\boldsymbol{a}^-$:
$[\boldsymbol{a}^+, \boldsymbol{a}^-] = -[\boldsymbol{a}^-, \boldsymbol{a}^+] = -1$.
So, $[\boldsymbol{a}^+, \boldsymbol{N}] = -\boldsymbol{a}^+$.
At the top of page 334, Eq.(10.25), the commutators for the harmonic oscillator are listed. Then it is shown how the raising operator $\boldsymbol{a}^+$ works. A kind of induction procedure is used.
The method of induction (Chisholm and Morris): assume the result to be true for one value of n, say $n = m$, show that the expression for $m + 1$ is the same as the expression for

$n = m$. Then it is true for all integral values of n.
Susskind writes, *by definition,* $\boldsymbol{N}|n\rangle = n|n\rangle$.
So, do we find $\boldsymbol{N}|(n+1)\rangle = (n+1)|(n+1)\rangle$?
Well, what we see is a new vector $\boldsymbol{a}^+|n\rangle = |n+1\rangle$, Eq.(10.27.
Is this the same as saying: the expression for $n+1$ is the same as for n? What we can say is, when we substitute for $\boldsymbol{a}^+|n\rangle$, the vector $|n+1\rangle$ in Eq.(10.26), we find
$\boldsymbol{N}|(n+1)\rangle = (n+1)|(n+1)\rangle$.
The preceding expression is similar to $\boldsymbol{N}|n\rangle = n|n\rangle$.
I am not convinced that this is a rigorous proof. Well, at least it be considered as an application of the definition,
$\boldsymbol{N}|n\rangle = n|n\rangle$.

Exercise with raising(creation) operator

Again, a new vector is obtained by operating $\boldsymbol{a}^+$ on the vector $|(n+1)\rangle$. Then with the same procedure given at the bottom of page 334:
$\boldsymbol{N}(\boldsymbol{a}^+|n+1\rangle = (\boldsymbol{a}^+\boldsymbol{N} - (\boldsymbol{a}^+\boldsymbol{N} - \boldsymbol{N}\boldsymbol{a}^+))|n+1\rangle =$
$= \big(\boldsymbol{a}^+\boldsymbol{N} - (-\boldsymbol{a}^+)\big)|n+1\rangle = (\boldsymbol{a}^+\boldsymbol{N} + \boldsymbol{a}^+)|n+1\rangle = \boldsymbol{a}^+\boldsymbol{N}|n+1\rangle + \boldsymbol{a}^+|n+1\rangle.$
With
$\boldsymbol{N}|(n+1)\rangle = (n+1)|(n+1)\rangle$**:**
$\boldsymbol{N}(\boldsymbol{a}^+|n+1\rangle) = \boldsymbol{a}^+(n+1)|n+1\rangle + \boldsymbol{a}^+|n+1\rangle = (n+2)\boldsymbol{a}^+|n+1\rangle.$
Like Eq.(10.27):
$\boldsymbol{a}^+|n+1\rangle = |n+2\rangle$
and,
$\boldsymbol{N}|(n+2)\rangle = (n+2)|(n+2)\rangle.$

Now the lowering operator.
We start again with $\boldsymbol{N}|n\rangle = n|n\rangle$, and introduce a new vector $\boldsymbol{a}^-|n\rangle$. With the procedure at the bottom of page 334 we have:
$\boldsymbol{N}(\boldsymbol{a}^-|n\rangle) = \big(\boldsymbol{a}^-\boldsymbol{N} - (\boldsymbol{a}^-\boldsymbol{N} - \boldsymbol{N}\boldsymbol{a}^-)\big)|n\rangle = \boldsymbol{a}^-\boldsymbol{N}|n\rangle - \boldsymbol{a}^-|n\rangle.$
With $\boldsymbol{N}|n\rangle = n|n\rangle$:
$\boldsymbol{N}(\boldsymbol{a}^-|n\rangle) = \boldsymbol{a}^- n|n\rangle - \boldsymbol{a}^-|n\rangle = (n-1)\boldsymbol{a}^-|n\rangle.$
By the induction procedure of page 334, we obtain Eq.(10.28):
$\boldsymbol{a}^-|\boldsymbol{n}\rangle = |n-1\rangle$
and
$\boldsymbol{N}|(n-1)\rangle = (n-1)|(n-1)\rangle.$
Susskind writes at the bottom of page 335: *"What about the lowering operator? Not surprisingly, we find that* $\boldsymbol{a}^-|n\rangle$ *produces an eigenvector whose eigenvalue is one unit lower:* $\boldsymbol{a}^-|n\rangle = |(n-1)\rangle$."
As mentioned by Susskind this lowering cannot go on and the downward sequence must end. To this end the ground state as the lowest energy state $|0\rangle$ must be introduced. A sort of Archimedes 'lever.
Susskind(page): "*Being the lowest energy state,* $|0\rangle$ *is the ground state, and its energy is:* $E_0 = \frac{\omega\hbar}{2}$. *(*$|0\rangle$*) is an eigenvector of* $\boldsymbol{N}$ *with an eigenvalue* 0". This statement by Susskind is not quite clear to me. The energy of the ground-state $|0\rangle$ is: $E_0 = \frac{\omega\hbar}{2}$.

So, consequently $E_0 = \frac{\omega\hbar}{2}$ is an eigenvalue of $|0\rangle$.
However, following the statement of Susskind, $|0\rangle$ is an eigenvector of $\boldsymbol{N}$ with an eigenvalue of zero. I think we learned from Lecture 10.5 :
$E_0 = \frac{\omega\hbar}{2}$ represents the ground-state energy and is an eigenvalue of $\boldsymbol{H}$**.** This is illustrated by $\boldsymbol{H} - \frac{\omega\hbar}{2}\boldsymbol{N} = \frac{\omega\hbar}{2}$. I prefer instead of $\boldsymbol{a}^-|0\rangle = 0$, Eq. 10.29, $\boldsymbol{N}\boldsymbol{a}^-|0\rangle = 0|0\rangle$. Then it is clear $E_0 = \frac{\omega\hbar}{2}$, being an eigenvalue of $\boldsymbol{H}$**.**
On page 336 Susskind writes: "*We often say that the ground-state is annihilated by* $\boldsymbol{a}^-$*".*
Well, I just learned that the ground-state is the lowest allowable and possible value of E. So, it appears to me that the effect of the operator $\boldsymbol{a}^-$ is *annihilated.* May be *annihilated* means something special.
Also operating $\boldsymbol{a}^-$ on $\psi_0(x)$ resulted into the non-zero ground-state wave function $\exp\left(-\frac{\omega}{2\hbar}x^2\right)$. The ground-state is certainly not annihilated. This came to me as a certain wake up call. The operator for finding the observables is the Hamiltonian which can be written as Eq. 10.22:
$\boldsymbol{H} = \omega\hbar(\boldsymbol{a}^+\boldsymbol{a}^- + \frac{1}{2})$.
So, the raising and lowering operators are acting together.
I think the raising and lowering operators with the newly defined vectors $\boldsymbol{a}^+|n\rangle$, and $\boldsymbol{a}^-|n\rangle$ are used to show the eigenvalues of the Hamiltonian change with integer steps. So, in this way you can find the eigenvalue spectrum. These new vectors illustrate the way the eigen vector spectrum can be found.
On page 335 in Fig. 10.2 a 'ladder' is shown. This ladder or the raising and lowering operators are developed by Dirac. The combined operators are sometimes called the ladder operator.
On the internet you can find nice pictures of the quantum harmonic oscillator.
On page 336 Susskind also writes: "*It allowed us to find the entire spectrum of harmonic oscillator energy levels without solving a single difficult equation*".
Well, almost. Susskind gave the solution for the ground state, Eq. (10.15) :
$\psi(x) = e^{-\frac{\omega}{2\hbar}x^2}$ (not normalized and not in the real world) and derived the ground state energy $E_0 = \frac{\omega\hbar}{2}$, Eq. (10.16) and (10.30) The normalized wave function of the ground-state is:
$\psi_0 = (\frac{\omega m}{\hbar\pi})^{1/4} e^{-\frac{\omega m}{2\hbar}y^2}$.

10.7 Back to Wave Functions

Susskind mentioned operator algebra to be rather abstract. To show the usefulness, Eq.(10.29) is rewritten in terms of the position and momentum operators, page 337.
On page 339 the exited state of the harmonic oscillator has been derived, using the raising operator:
$\psi_1 = 2i\omega x e^{-\frac{\omega}{2\hbar}x^2}$ (not in the real world),
without the normalization factor for the ground-state in the x-space.
What does the expression for the exited state looks like in the real world? Well, the proof of the pudding is in the eating. We follow the approach of Susskind on page 339 and write:

$\psi_1(y) = (-\frac{i\hbar}{\sqrt{m}}\frac{\partial}{\partial y} + i\omega\sqrt{m}y)(\frac{\omega m}{\hbar\pi})^{1/4}e^{-\frac{\omega m}{2\hbar}y^2}$,
where we used the ground-state ψ_0 in the real world.
This leads to $\psi_1(y) = 2i\omega\sqrt{m}y(\frac{\omega m}{\pi\hbar})^{1/4}\exp(-\frac{m\omega}{2\hbar}y^2)$.
This exited wave function is not normalized. To correct this, we must multiply the raising operator by the constant $-i/\sqrt{2\omega\hbar}$.
This indeed effects the numerical coefficient, the amplitude. By plugging this constant back into the exited state $\psi_1(y)$, this state is normalized. Why bother? Well, reading and comparing the results of Mahan on the harmonic oscillator you could get confused. In addition, you can call it a numerical coefficient. However, it is part of the amplitude of the eigenfunction. We learned the amplitude to be an important quantity in doing experiments: the probability of finding a particle in the real world. So, for completeness I give the exited wave function:
$\psi_1(y) = (\frac{2}{\pi})^{5/4}(\frac{m\omega}{2\hbar})^{3/4}\,\mathrm{y}\,\exp(-\frac{m\omega}{2\hbar}y^2)$,
or:
$\psi_1(y) = \frac{2}{\pi}\left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{2}}y\psi_0(y)$.
On page 341 is written: "*The ground-state eigenfunction* $e^{-\frac{\omega}{2}x^2}$ *……..*". A typo. The ground-state is: $\exp\left(-\frac{\omega}{2\hbar}x^2\right)$. There Susskind mentioned the Hermite polynomials for the eigenfunctions of the Schrödinger equation.
Mahan, page 30 and 31 presented the eigenfunction solution with Hermite polynomials, Eqs.(2.108)-(2.114).
First I summarize the results of Susskind for the ground state and the first two exited states:
$\psi_0 \propto e^{-\frac{\omega x^2}{2\hbar}}$, $\psi_1 \propto xe^{-\frac{\omega x^2}{2\hbar}}$ and $\psi_2 \propto (x^2 - \frac{\hbar}{2\omega})e^{-\frac{\omega x^2}{2\hbar}}$.
Now Mahan:
$\psi_0 \propto e^{-\frac{m\omega x^2}{2\hbar}}$, $\psi_1 \propto xe^{-\frac{m\omega x^2}{2\hbar}}$ and $\psi_2 \propto (x^2 - \frac{\hbar}{2m\omega})e^{-\frac{m\omega x^2}{2\hbar}}$.
Mahan has the mass m included. This is explained by Susskind by the definition where m is absorbed in the new x on page 314.

10.8 The Importance of Quantization

In the last Lecture, Susskind painted a picture for future lectures, especially quantum field theory. Susskind showed some examples like the oscillating electric and magnetic fields, Figure 10.3.
The importance of frequency is mentioned: *"…the frequency determines the quantum energy of the oscillator."*
The elementary particle *photon* and its energy is introduced.
Susskind: "*In the end, it all goes back to the harmonic oscillator*".

Epilogue

We learned about the Schrödinger equation. An equation describing the steady flow of matter waves. Born reinterpreted the wave function as probabilities. This brought

Schrödinger's equation in line with Heisenberg's indeterminacy. I do not know whether Schrödinger appreciated that (Halpern). Reading Cox and Forshaw, I got the impression that Schrödinger was not amused by the Born interpretation.
I leave Volume II of the Theoretical Minimum Series. Let us look forward to the next volume on Quantum Mechanics.
Remark: The next volume I laid my hands on is about *Special Relativity and Classical Field Theory,* 2017.

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