Adaptation to Climate via Morphological Change. Optimizing Heat Exchange

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www.leennoordzij.me

dr.l.noordzij@leennoordzij.nl

Inhoudsopgave

Prologue, a Boreal Adaptation to Climate Change?

In the Netherlands, the height of people in the south is 3 cm smaller than the height of people in the north. The Netherlands measures about 250 kilometre from north to south. The mean temperature in the north is a couple of °C lower than in the south.
Ryding et al: "*The ecophysiological significance of heat exchange becomes more apparent when looking at global trends of body size and shape, as formalised by two biogeographical rules: Bergmann's and Allen's rules. These focus on gradients in body size and appendage size, respectively. Bergmann's rule states that higher latitudes correlates with larger body sizes: In the cooler climates found at higher latitudes, the reduced body surface area/volume ratio of larger bodies enables animals to retain heat more effectively.*"

§ 1 Introduction

The isoperimetric inequality is analysed to some extend in, a.o., Noordzij(1). The idea to analyse the isoperimetric inequality was stimulated by a paper of Luttwak describing the history of building fortifications. It is about to find the maximum area for a given perimeter. Is there a 3-D equivalent? There is. It is not about building fortifications, it is about metabolism of vertebrates, Noordzij (2). The relation between surface and volume of a vertebrate is investigated. The 3-D equivalent with the isoperimetric inequality is about to find the maximum volume for a given surface. I denoted this the *isoepifaeic inequality*.
This inequality can also be used in the study of climate change and evolution. In The Economist the subject matter is presented based on a study in *Trends in Ecology & Evolution*, by Ryding, et all. It is about adapting to climate change. The paper concluded climate change already to alter the bodies of animal species. To improve heat exchange with the environment, this adaptation gives rise to an increasing surface for a given volume.
First, in §2, we investigate the maximum volume, mass, for a given surface. Next, in §3, we study the possibilities to adjust a given surface to adapt to rising temperatures: climate change. Adaptation, according to the research of Ryding, et all, is about altering the bodies of animal species by increasing the dimensions of appendages. In this case a rising temperature of the environment is assumed and the adaptation of the vertebrate by increasing its surface for a given volume. Or, by decreasing the volume for a given surface in order to improve heat exchange with the environment.
In §4, the primal body and the appendages are optimized.
The article in The Economist opened with the statement: "*For humans, adapting to climate change will mostly be a matter of technology*." For all human beings?

§ 2 The Block, the Cylinder, the Cone, the Pyramid and Heat Exchange with the Environment

In this paragraph we examine various geometrical structures
Heat exchange optimization with the environment is investigated of the aforementioned geometrical structures. This is done for a given volume of those structures.

§ 2.1 A Block with Surface S . Optimizing Heat Exchange

In Figure 2.1 below, the dimensions of the block are given.

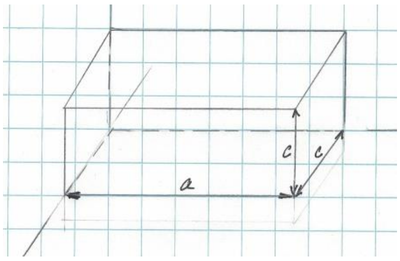


Figure 2.1 A Block wit Surface S

The surface
$S = 2c^2 + 4ac$.
Then
$a = \frac{S-2c^2}{4c}$.
$V = a \cdot c^2 = \frac{S \cdot c - 2c^3}{4}$.
Assume a given volume V of the block. What do we find for the optimized surface S_{block} ?
The volume:
$V = a \cdot c^2 \Longrightarrow a = \frac{V}{c^2}$.
The surface
$S_{block} = 2c^2 + 4ac = 2c^2 + \frac{4V}{c}$.
$\frac{dS_{block}}{dc} = 0 = 4c - \frac{4V}{c^2} \Longrightarrow V = c^3 \Longrightarrow$ a cube. (notice $V = a \cdot c^2 \Longrightarrow a = c$).
Furthermore
$\frac{d^2 S_{block}}{dc^2} = 4 + \frac{8V}{c^3} > 0$, a minimum area S for a cube.

To make a plot of $S_{block} (= 2c^2 + 4ac)$, I make the function S_{block} dimensionless:
$c' = \frac{c}{V^{1/3}}$, and $S'_{block} = \frac{S_{block}}{V^{2/3}}$.
After dropping the primes :
$S_{block} = 2c^2 + \frac{4}{c}$.
In the Figure[1] below: $y \equiv S_{block}$, and $x \equiv c$.

[1] With the WolframAlpha App

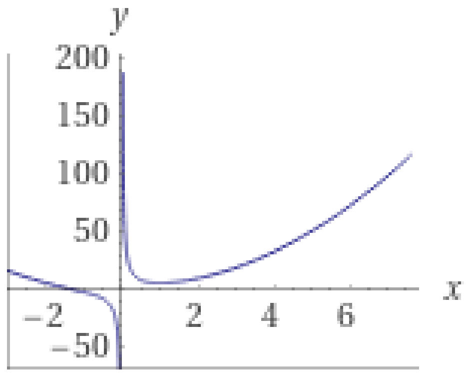


Figure 2.2 The surface of the block S(≡y) as a function of c(≡x) for a given volume of the block

The minimum value of y is at $x = 1$, a cube.
This plot illustrates, for a given volume, the decrease of c , left of the minimum, is more effective in exchanging heat with the environment than an crease of c, right of the minimum.
With this result, it becomes clear that by changing the cube(the volume at the minimum value of S_{block}) into a block with the same volume($V = a \cdot c^2$), Figure 2.1, the area of the body increases for the given mass(volume). Consequently, the heat transfer with the environment increases.
As mentioned in the introduction, adaptation is about increasing the surface of the appendages. In this way the heat exchanging area is enlarged relative to the volume(mass). What will the geometry of the appendages be to make the heat exchange efficient? WE investigate appendages in the next paragraphs.

§ 2.2 The Cone with Surface S, the Beak. Optimizing Heat Exchange

With this example we come close to one of the examples of Ryding et all: the beak of a parrot considered to be an appendage.

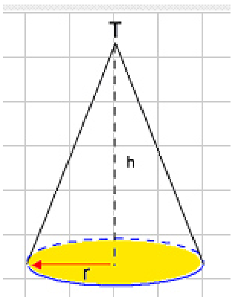


Figure 2.3 The Cone

In Figure 2.3 above an example of the cone is presented.
We investigate the optimization of heat exchange with the environment.
The area for the surface, the conical surface and the surface of the ground circle:
$S = \pi r R + \pi r^2$,
where the slant height of the cone $R = \sqrt{r^2 + h^2}$.
It is about heat exchange with the environment. For that reason, I investigate the surface

πrR and exclude πr^2. The area of the basis of the cone does not play a direct role in the process of heat exchange with the environment. The basis does play a role in the transport of blood to and from the beak.

So, we have

$S = \pi rR,$

and

$$V_{cone} = \frac{\pi}{3}\sqrt{R^2 - r^2} \cdot r^2 \Longrightarrow R = \sqrt{(\frac{3V_{cone}}{\pi r^2})^2 + r^2}.$$

Then,

$$S = \pi rR = \sqrt{\frac{9V_{cone}^2}{r^2} + \pi^2 r^4}\ .$$

For further analysis I make the function $S = \sqrt{\frac{9V_{cone}^2}{r^2} + \pi^2 r^4}$, dimensionless with:

$r' = \frac{r}{V_{cone}^{1/3}}$, and $S' = \frac{S}{V_{cone}^{2/3}}$.

After dropping the primes we have

$$S = \sqrt{\frac{9}{r^2} + \pi^2 r^4}\ .$$

A plot, Figure 2.4, of this function of S, where $y = S$ and $x = r$.

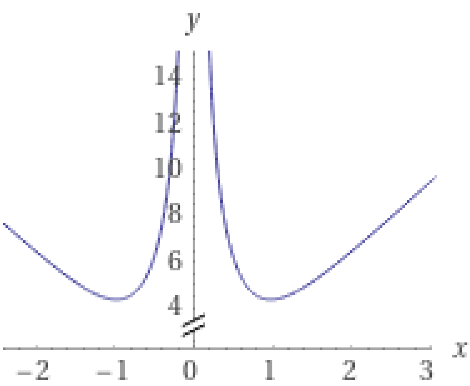


Figure 2.4 The surface of the cone S(≡y) as a function of r(≡x) for a given volume of the cone

This plot illustrates, for a given volume, the decrease of r , at the left of the minimum, is more effective in exchanging heat with the environment than an increase of r, at the right of the minimum

The minimum value for S is obtained at $r^3 = \frac{3}{\pi\sqrt{2}}$, dimensionless.

$\frac{d^2S}{dr^2} > 0$, a minimum indeed.

Hence, the most efficient heat exchange is obtained by decreasing r and consequently increasing R. Is that observed in the real world? Mating and food gathering can play an equal key role. Consequently, the evolutionary route can be different and an increase of r can be more important in that case.

§ 2.3 The Cylinder with Surface S, the Limb. Optimizing Heat Exchange

Citing the paper of Ryding et all, : "*…larger appendages may be advantageous in warmer climates.*" Let us investigate the subject matter. It is about the increase of appendages like limbs relative to body size.

The surface, S, of a cylinder with length L and radius r:
$S = 2\pi rL + 2\pi r^2$.
So
$L = \frac{S}{2\pi r} - r$.
We investigate the optimization of heat exchange with the environment.
The area for the surface
$S = 2\pi rL + \pi r^2$.
It is about heat exchange with the environment, for that reason one πr^2 connecting the cylinder(limp) with the principal body is deleted in the above expression for the surface S. This part of the surface of the cylinder does not play a direct role in the process of heat exchange with the environment. This part does play a role in the transport of blood to and from the limb.
The volume of the limb
$V_{cyl} = \pi r^2 L \Longrightarrow L = \frac{V_{cyl}}{\pi r^2}$.
Plug this expression for L into the expression for the surface:
$S = 2\pi rL + \pi r^2 = \frac{2V_{cyl}}{r} + \pi r^2$.
I make the preceding expression dimensionless:
$r' = \frac{r}{V_{cyl}^{1/3}}$, and $S' = \frac{S}{V_{cyl}^{2/3}}$.
After dropping the primes we have
$S = \frac{2}{r} + \pi r^2$.
Two plots, Figure 2.5, of this function this function of S, where $y \equiv S$ and $x \equiv r$.

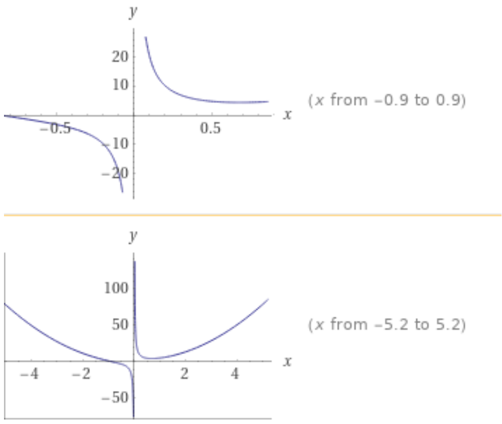


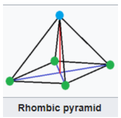
Figure 2.5 The surface of the cylinder S(≡y) as a function of r(≡x)

In the lower plot the minimum is demonstrated.
The minimum:
$\frac{dS}{dr} = 0 \Longrightarrow r = (\frac{1}{\pi})^{1/3}$.

$\frac{d^2S}{dr^2} = \frac{4}{r^3} + 2\pi > 0$, a minimum.
Hence, the most efficient heat exchange is obtained by decreasing r, at the left of the minimum and consequently increasing L for a given volume V_{cyl}. Is that observed in the real world?

§ 2.4 The Pyramid with a Rhombic shaped Base and Surface S. Another Beak, Optimizing Heat Exchange

In Figure 2.6 below the rhombic pyramid is illustrated (see insert).



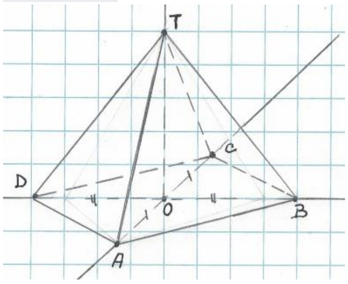


Figure 2.6 The Rhombic Pyramid

Top view:

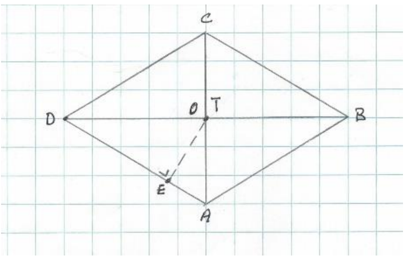


Figure 2.7 Top view of Figure 2.6

$ABCD$ is the rhombic base.
Denote $DB \to w, AC \to h, OE \to x, EA \to y, ET \to z$ and $OT \to t$
In the above top view TE/OE represent the perpendicular of the ΔTAD and ΔOAD respectively.

$x^2 = \frac{1}{4} \cdot h^2 - y^2,$

and

$x^2 = \frac{1}{4} \cdot w^2 - (\frac{1}{2}\sqrt{h^2 + w^2} - y)^2.$

Then,

$y = \frac{1}{2}\frac{h^2}{\sqrt{h^2+w^2}},$

and

$x = \frac{1}{2}\frac{hw}{\sqrt{h^2+w^2}}.$

Furthermore $z = \sqrt{x^2 + t^2}$.

With $x = \frac{1}{2}\frac{hw}{\sqrt{h^2+w^2}},$

area $\Delta TAD \rightarrow \frac{1}{2}z \cdot |DA| = \frac{1}{2}\sqrt{x^2+t^2} \cdot \frac{1}{2}\sqrt{h^2+w^2} = \frac{1}{4}\sqrt{t^2(h^2+w^2)+\frac{1}{4}h^2w^2}.$

The area of the pyramid is

$$S = \frac{1}{2} \cdot hw + \sqrt{t^2(h^2+w^2)+\frac{1}{4}h^2w^2}\ .$$

The volume of the pyramid is

$$V = \frac{1}{3} \cdot \frac{1}{2} \cdot\ hw \cdot t.$$

Heat exchange is about the largest area for a given volume.

Similarly to the other appendages as presented in the preceding paragraphs, the area of the basis of the pyramid does not contribute in the heat exchange.

Consequently S to be investigated is

$$S = \sqrt{t^2(h^2+w^2)+\frac{1}{4}h^2w^2}$$

In the expression for the surface plug, with $V = \frac{1}{3} \cdot \frac{1}{2} \cdot\ hw \cdot t$,

$t = \frac{6V}{hw}$ into

$$S = \sqrt{t^2(h^2+w^2)+\frac{1}{4}h^2w^2} \Longrightarrow S = \sqrt{(\frac{6V}{hw})^2(h^2+w^2)+\frac{1}{4}h^2w^2}.$$

We make the preceding expression dimensionless with h$= h'V_p^{1/3},\ w = w'V_p^{1/3}$, and

$$S' = \frac{S}{V_p^{2/3}}$$

dropping the primes:

$$S = S(h,w) = \sqrt{(\frac{6}{hw})^2(h^2+w^2)+\frac{1}{4}h^2w^2}.$$

To investigate stationary point(s) the equations $\frac{\partial S}{\partial h} = 0$ and $\frac{\partial S}{\partial w} = 0$ are solved.

Leaving out the details, the result is $h = w$.

Is the stationary point a maximum a maximum? To visualize this $S(h,1)$ and $S(1,w)$. Are plotted.

In Figure 2.8, $S(h,1)$ is plotted

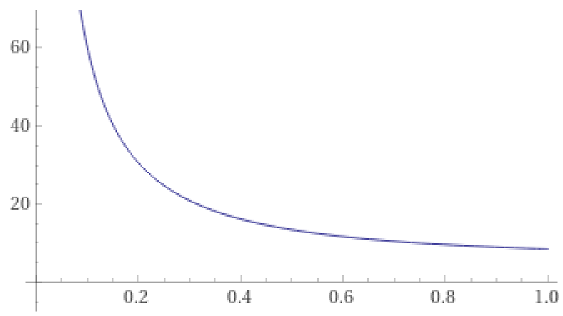


Figure 2.8 S(h,w) is plotted with w=1

The above Figure represents also $S(h,w)$ with $h = 1$.

Hence, we may conclude to have a minimum at $h = w = 1$.

Consequently, to adapt to climate change and improve heat exchange, h and or w need to decrease. The height of the pyramid $t = \frac{6V}{hw}$ will increase.

§ 3 The Primal Body and Optimizing the Appendages

It is about warm-blooded animals.
The Economist: " *Climate change is already altering the bodies of many animal species, giving them bigger beaks, limbs and ears."*
Knowing this, modelling of metabolism can be of helpful. The changes do help the animals to increase the heat exchange with the environment. The key issue here is increasing the surface area relative to its body volume.
In this paragraph we investigate heat exchange between a geometrical structure consisting of a primal or main body and appendages and the environment. The dimensions of the primal body are given, and the changes of the appendage are investigated.
In the following paragraphs, I assume the primal body to participate in the heat exchange. If not, the analysis of § 2 can be used.

§ 3.1 The Sphere as a Primal Body and the Limbs

The Economist: "*Another evolutionary rule-of-thumb, Bergmann's rule holds that animals in hotter places tend to have smaller bodies, another way to boost the ratio between surface area and volume."*
In studying metabolism, the primal body as the bigger part of the overall body size of an animal is usually approximated by a sphere.
The ratio of surface area and volume of a sphere:

$$ratio = \frac{4\pi r^2}{\frac{4}{3}\pi r^3} = \frac{3}{r}.$$

Next reduce the radius with a small quantity Δr, with $\Delta \ll 1$.
Plug this into the preceding expression

$$ratio = \frac{3\pi}{r} \Longrightarrow \frac{3\pi}{r(1-\Delta)} = \frac{3\pi}{r}[1 + \Delta + O(\Delta^2)] \Longrightarrow ratio = \frac{3\pi}{r}(1 + \Delta).$$

Obviously, the $ratio$ increases by a factor $(1 + \Delta)$ and *the ratio between surface area and volume is boosted.* In this way heat exchange with the environment is enhanced.

In the following we analyse the addition of limbs.
- **First, let us introduce four appendages**: the limbs with a cylindrical geometry. Can heat exchange be enhanced by, e.g., increasing the length of the cylinders?

$$ratio_B = \frac{4\pi r_{sh}^2 + 8\pi r_{cl} L + 4\pi r_{cl}^2}{\frac{4}{3}\pi r_{sh}^3 + 4\pi r_{cl}^2 L} \equiv \frac{S_B}{V_B},$$

where the suffix B denotes the base , and the suffixes sh and cl denote the sphere and cylinder respectively. Furthermore, I assume the cylindrical limbs to reduce the area of the sphere with $4\pi r_{cl}^2$.
Consequently,

$$ratio_B = \frac{4\pi r_{sh}^2 + 8\pi r_{cl} L}{\frac{4}{3}\pi r_{sh}^3 + 4\pi r_{cl}^2 L} \equiv \frac{S_B}{V_B}.[2]$$

[2] For $r_{cl} \to 0$, and $L \to 0$, the surface to volume ratio of the sphere is found as it should be.

Then, we increase the length of the limbs by ΔL , with $\Delta \ll 1$.[3]

Plug $L \Longrightarrow L+\Delta L$ into the ratio $ratio_B$ giving the new ratio $ratio_{BN}$

$$ratio_{B\Delta L}=\frac{S_B}{V_B}\frac{(1+\frac{8\pi r_{cl}\Delta L}{S_B})}{(1+\frac{4\pi r_{cl}^2\Delta L}{V_B})}\cong\frac{S_B}{V_B}\left(1+\frac{8\pi r_{cl}\Delta L}{S_B}\right)\left(1-\frac{4\pi r_{cl}^2\Delta L}{V_B}\right)=\frac{S_B}{V_B}\left(1+\frac{8\pi r_{cl}\Delta L}{S_B}-\frac{4\pi r_{cl}^2\Delta L}{V_B}\right)=$$

$$=ratio_B\left(1+\frac{8\pi r_{cl}\Delta L}{S_B}-\frac{4\pi r_{cl}^2\Delta L}{V_B}\right),$$

up to $O(\Delta^2)$.

From the preceding expression we conclude

$ratio_{B\Delta L} > ratio_B$,

when

$$\frac{8\pi r_{cl}\Delta L}{S_B}-\frac{4\pi r_{cl}^2\Delta L}{V_B}>0,$$

or

$$\frac{2}{S_B}-\frac{r_{cl}}{V_B}>0 \Longrightarrow 2V_B>r_{cl}S_B.$$

Plug the expressions for V_B and S_B into the preceding expression

$$2\left(\frac{4}{3}\pi r_{sh}^3+4\pi r_{cl}^2L\right)>r_{cl}(4\pi r_{sh}^2+8\pi r_{cl}L) \Longrightarrow \frac{2}{3}r_{sh}^3>r_{cl}r_{sh}^2 \Longrightarrow$$

$$\Longrightarrow \frac{2}{3}>\frac{r_{cl}}{r_{sh}}.$$

In the real world in most cases $\frac{r_{cl}}{r_{sh}}<O(10^{-1})$

Consequently, by enlargement ΔL of the (four) appendages(limbs) heat exchange is enhanced under the condition $\frac{2}{3}>\frac{r_{cl}}{r_{sh}}$.

Note : in Appendix A1 I analysed the $ratio_B$ as a function of two variables. There, as a result of the optimization process

$$\frac{2}{3}=\frac{r_{cl}}{r_{sh}},$$

is obtained. Then, we are allowed to conclude for $\frac{r_{cl}}{r_{sh}}<\frac{2}{3}$, heat exchange is improved.

Now, we assume L to be a constant and investigate a change Δr_{cl}.

With,

$$ratio_B=\frac{4\pi r_{sh}^2+8\pi r_{cl}L;}{\frac{4}{3}\pi r_{sh}^3+4\pi r_{cl}^2L}\equiv\frac{S_B}{V_B}.$$

Then, we increase the radius of the limbs by Δr_{cl} , with $\Delta \ll 1$.

$$ratio_{B\Delta r_{cl}}=\frac{S_B}{V_B}\frac{(1+\frac{8\pi L\Delta r_{cl}}{S_B})}{(1+\frac{8\pi r_{cl}L\Delta r_{cl}}{V_B})}\cong ratio_B\left(1+\frac{8\pi L\Delta r_{cl}}{S_B}-\frac{8\pi r_{cl}L\Delta r_{cl}}{V_B}\right),$$

up to $O(\Delta^2)$.

From the preceding expression we conclude

$ratio_{B\Delta r_{cl}} > ratio_B$,

when

$$\frac{8\pi(L+r_{cl})\Delta r_{cl}}{S_B}>\frac{8\pi r_{cl}L\Delta r_{cl}}{V_B},$$

or

[3] Increasing just the limbs by ΔL, r_{cl} is treated to be a constant.

$\frac{L+r_{cl}}{S_B} > \frac{r_{cl}L}{V_B} \Longrightarrow r_{cl}L \cdot S_B < L \cdot V_B.$

Plug the expressions for V_B and S_B into the preceding expression

$r_{cl}L(r_{sh}^2 + 2r_{cl}L) < (\frac{1}{3}r_{sh}^3 + r_{cl}^2 L)L.$[4] For $L \neq 0$: $r_{cl}(r_{sh}^2 + r_{cl}L) < \frac{1}{3}r_{sh}^3$.

- **Next, let us introduce two appendages**: the limbs with a cylindrical geometry. Can heat exchange be enhanced by, e.g., increasing the length of the cylinders?[5]

I use the same approximation as used for the four appendages.

$$ratio_B = \frac{4\pi r_{sh}^2 + 4\pi r_{cl}L}{\frac{4}{3}\pi r_{sh}^3 + 2\pi r_{cl}^2 L} \equiv \frac{S_B}{V_B},$$

where the suffix B denotes the base , and the suffixes sh and cl denote the sphere and cylinder respectively. Furthermore, $\frac{r_{cl}}{r_{sh}} \ll 1$

Then, we increase the length of the limbs by ΔL , with $\Delta \ll 1$.

Plug this increase into the ratio $ratio_{SB}$ giving the new ratio $ratio_{SBN}$

$$ratio_{B\Delta L} = \frac{S_B}{V_B}\frac{(1+\frac{4\pi r_{cl}\Delta L}{S_B})}{(1+\frac{2\pi r_{cl}^2\Delta L}{V_B})} \cong \frac{S_B}{V_B}\left(1 + \frac{4\pi r_{cl}\Delta L}{S_B}\right)\left(1 - \frac{2\pi r_{cl}^2\Delta L}{V_B}\right) = \frac{S_B}{V_B}\left(1 + \frac{4\pi r_{cl}\Delta L}{S_B} - \frac{2\pi r_{cl}^2\Delta L}{V_B}\right) =$$

$$= ratio_B\left(1 + \frac{4\pi r_{cl}\Delta L}{S_B} - \frac{2\pi r_{cl}^2\Delta L}{V_B}\right),$$

up to $O(\Delta^2)$.

From the preceding expression we conclude

$ratio_{B\Delta l} > ratio_B$,

when

$$\frac{4\pi r_{cl}\Delta L}{S_B} - \frac{2\pi r_{cl}^2\Delta L}{V_B} > 0,$$

or

$$\frac{2}{S_B} - \frac{r_{cl}}{V_B} > 0 \Longrightarrow 2V_B > r_{cl}S_B.$$

Plug the expressions for V_B and S_B into the preceding expression

$$2\left(\frac{4}{3}\pi r_{sh}^3 + 2\pi r_{cl}^2 L\right) > r_{cl}(4\pi r_{sh}^2 + 4\pi r_{cl}L) \Longrightarrow \frac{2}{3}r_{sh}^3 > r_{cl}r_{sh}^2 \Longrightarrow$$

$$\Longrightarrow \frac{2}{3} > \frac{r_{cl}}{r_{sh}}.$$

So,

$\frac{r_{cl}}{r_{sh}} < \frac{2}{3}$. In the real world in most cases $\frac{r_{cl}}{r_{sh}} < \frac{2}{3}$.

Consequently, by enlargement of the (two) appendages(limbs) heat exchange is enhanced.

Note: a “biological” limit is the area of halve the sphere to be larger than the area cut out of the sphere by the limbs. Hence, $2\pi r_{sh}^2 > 4\pi r_{cl}^2 \Longrightarrow \frac{r_{cl}}{r_{sh}} < \sqrt{\frac{1}{2}}$, in the case of four limbs. In the case of two limbs: $\frac{r_{cl}}{r_{sh}} < 1$.

[4] Make this expression dimensionless similarly to what we did in Appendix A1 and the same result is obtained: $\frac{1}{3}y = xy(1 + xy)$, at the saddle point $(\frac{2}{3}, -\frac{3}{4})$. A meaningless stationary point.

[5] Consequently , again I treated the radius of the cylinders to be a constant. In general, $ratio_B$ is a function of two independent variables(See Appendix) The dimensions of the primal body are constant.

§ 3.2 The Cylinder as a Primal Body and the Limbs

In § 3.1 we approximated the primal body by a sphere. It is not unthinkable to approximate the primal body by a cylinder.

The ratio of surface area and volume of a cylinder:

$$ratio = \frac{2\pi rL+2\pi r^2}{\pi r^2 L} = \frac{2L+2r}{rL}.$$

Reduce the radius of the primal body with a small quantity Δr, or the length of the cylinder representing the principal body is reduced with a small quantity ΔL , $\Delta \ll 1$.

Plug $r(1-\Delta)$ into the preceding expression

$$ratio = \frac{2L+2r}{rL} \Longrightarrow \frac{2L+2r(1-\Delta)}{r(1-\Delta)L} \approx \frac{2L+2r}{rL}\left[1+\frac{\Delta}{1+\frac{r}{L}}+O(\Delta^2)\right] \Longrightarrow ratio = \frac{2L+2r}{rL}(1+\frac{\Delta}{1+\frac{r}{L}}).$$

Obviously, the $ratio$ increases by a factor $(1+\frac{\Delta}{1+\frac{r}{L}})$ and *the ratio between surface area and volume is boosted.* In this way heat exchange with the environment is enhanced. However, by decreasing the radius r.

The other way to reduce the volume of the primal body is by decreasing L.

Plug $L(1-\Delta)$ into $ratio = \frac{2L+2r}{rL}$:

$$ratio = \frac{2L+2r}{rL} \Longrightarrow \frac{2L(1-\Delta)+2r}{rL(1-\Delta)} \approx \frac{2L+2r}{rL}\left[1+\frac{\Delta}{\frac{L}{r}+1}+O(\Delta^2)\right] \Longrightarrow ratio = \frac{2L+2r}{rL}(1+\frac{\Delta}{\frac{L}{r}+1}).$$

Also in this case *the ratio between surface area and volume is boosted.* However, with a much smaller effect, since $\frac{L}{r} > 1$.

In the following we analyse the addition of limbs to the cylindrical primal body.

- **First, let us introduce four appendages**: the limbs with a cylindrical geometry.

Furthermore, I assume the cylindrical limbs to reduce the area of the cylindrical primal body approximately with $4\pi r_{cl}^2$.

Can heat exchange be enhanced by, e.g., increasing the length of the cylinders(limbs)?

The base configuration:

$$ratio_B = \frac{2\pi rL+2\pi r^2+8\pi r_{cl}l}{\pi r^2 L+4\pi r_{cl}^2 l} \equiv \frac{S_B}{V_B},$$

where the suffix B denotes the base , and the suffix cl denote the cylinder(limb) . l is the length of the limb. Then, we increase the length of the limbs by Δl , with $\Delta \ll 1$.

Plug this increase into the ratio $ratio_B$ giving the new ratio $ratio_{BN}$

$$ratio_{B\Delta l} = \frac{S_B}{V_B}\frac{(1+\frac{8\pi r_{cl}\Delta l}{S_B})}{(1+\frac{4\pi r_{cl}^2\Delta l}{V_B})} \cong \frac{S_B}{V_B}\left(1+\frac{8\pi r_{cl}\Delta l}{S_B}\right)\left(1-\frac{4\pi r_{cl}^2\Delta l}{V_B}\right) = \frac{S_B}{V_B}\left(1+\frac{8\pi r_{cl}\Delta l}{S_B}-\frac{4\pi r_{cl}^2\Delta l}{V_B}\right) =$$

$$= ratio_B\left(1+\frac{8\pi r_{cl}\Delta l}{S_B}-\frac{4\pi r_{cl}^2\Delta l}{V_B}\right),$$

up to $O(\Delta^2)$.

From the preceding expression we conclude

$$ratio_{B\Delta l} > ratio_B,$$

when

$$\frac{8\pi r_{cl}\Delta l}{S_B}-\frac{4\pi r_{cl}^2\Delta l}{V_B} > 0,$$

or

$$\frac{2}{S_B}-\frac{r_{cl}}{V_B} > 0 \Longrightarrow 2V_B > r_{cl}S_B.$$

Plug the expressions for V_B and S_B into the preceding expression

$$2(\pi r^2 L + 4\pi r_{cl}^2 l) > r_{cl}(2\pi r L + 2\pi r^2 + 8\pi r_{cl} l) \Rightarrow r^2 L > r_{cl} r L + r_{cl} r^2 \Rightarrow$$

$$\Rightarrow 1 > \frac{r_{cl}}{r}(1 + \frac{r}{L}) .$$

For various values of $\frac{r}{L}$ [6], we could evaluate the inequality . Here, as a best guess, $\frac{r}{L}$ is chosen to be of order unity. Then, set $\frac{r}{L} = 1$ in the preceding expression

$$\frac{1}{2} > \frac{r_{cl}}{r} .$$

Furthermore, with $x = \frac{r_{cl}}{r}$, and $y = \frac{l}{r} \Rightarrow F = r \cdot ratio_B = \frac{1+2xy}{\frac{1}{4}+x^2 y}$. The analysis of $F(x, y)$ is similar to the analysis of Appendix 1.

In the real world in most cases $\frac{r_{cl}}{r} < 0.5$.

Consequently, by enlargement of the (four) appendages(limbs) heat exchange is enhanced.

- **Next, let us introduce two appendages:** the limbs with a cylindrical geometry. Can heat exchange be enhanced by, e.g., increasing the length of the cylinders(limbs)?

The base configuration:

$$ratio_B = \frac{2\pi r L + 2\pi r^2 + 4\pi r_{cl} l}{\pi r^2 L + 2\pi r_{cl}^2 l} \equiv \frac{S_B}{V_B} ,$$

where the suffix B denotes the base , and the suffix cl denote the cylinder(limb). l is the length of the limb.

Then, we increase the length of the limbs by Δl , with $\Delta \ll 1$. Furthermore, I assume the cylindrical limbs to reduce the area of the cylindrical primal body approximately with $4\pi r_{cl}^2$, $\frac{r_{cl}}{r} \ll 1$.

Plug this increase into the ratio $ratio_B$ giving the new ratio $ratio_{BN}$

$$ratio_{B\Delta l} = \frac{S_B}{V_B} \frac{(1+\frac{4\pi r_{cl}\Delta l}{S_B})}{(1+\frac{2\pi r_{cl}^2 \Delta l}{V_B})} \cong \frac{S_B}{V_B}\left(1 + \frac{4\pi r_{cl}\Delta l}{S_B}\right)\left(1 - \frac{2\pi r_{cl}^2 \Delta l}{V_B}\right) = \frac{S_B}{V_B}\left(1 + \frac{4\pi r_{cl}\Delta L}{S_B} - \frac{2\pi r_{cl}^2 \Delta l}{V_B}\right) =$$

$$= ratio_B \left(1 + \frac{4\pi r_{cl}\Delta l}{S_B} - \frac{2\pi r_{cl}^2 \Delta l}{V_B}\right),$$

up to $O(\Delta^2)$.

From the preceding expression we conclude

$$ratio_{B\Delta l} > ratio_B ,$$

when

$$\frac{4\pi r_{cl}\Delta l}{S_B} - \frac{2\pi r_{cl}^2 \Delta l}{V_B} > 0,$$

or

$$\frac{2}{S_B} - \frac{r_{cl}}{V_B} > 0 \Rightarrow 2V_B > r_{cl} S_B.$$

With the expressions for $V_B = \pi r^2 L + 2\pi r_{cl}^2 l$, and $S_B = 2\pi r L + 2\pi r^2 + 4\pi r_{cl} l$:

$$2V_B > r_{cl} S_B \Rightarrow 2(\pi r^2 L + 2\pi r_{cl}^2 l) > r_{cl}(2\pi r L + 2\pi r^2 + 4\pi r_{cl} l) \Rightarrow$$

$$\Rightarrow r^2 L > r_{cl} r L + r_{cl} r^2 \Rightarrow 1 > \frac{r_{cl}}{r}(1 + \frac{r}{L}) .$$

For various values of $\frac{r}{L}$, we can evaluate the threshold $\frac{r_{cl}}{r}$. Here, I again I choose $\frac{r}{L} = 1$.

Set $\frac{r}{L} = 1$ in the expression $1 > \frac{r_{cl}}{r}(1 + \frac{r}{L})$

[6] The ratio of the dimensions of the primal body.

$1 > 2\frac{r_{cl}}{r}$.
So,
$\frac{r_{cl}}{r} < \frac{1}{2}$.
In the real world in most cases $\frac{r_{cl}}{r} < \frac{1}{2}$.
The analysis of $r \cdot ratio_B$ is similar to the analysis of Appendix 1:
$F(x,y) = \frac{1+xy}{\frac{1}{4}+\frac{1}{2}x^2y}$.
Consequently, by enlargement of the (two) appendages(limbs) heat exchange is enhanced.
Note: a “biological” limit is the area of halve the cylinder representing the primal body to be larger than the area cut out of the sphere by the limbs. Hence, $\pi r L > 4\pi r_{cl}^2$. To be more precise, this constraint consists of two components:
- $\pi r > 4r_{cl}$,
- $L > 4r_{cl}$,
in the case of four limbs.
In the case of two limbs one constraint changes: $L > 2r_{cl}$.
Caveat, the constraints are approximations.

§ 3.3 The Sphere as a Primal Body and the Beak(Cone)

In § 3.1 we already showed by decreasing the radius of the sphere *the ratio between surface area and volume is boosted.* In this way heat exchange with the environment is enhanced.
In the foregoing paragraphs, I made an approximation of the effect of the combination of primal body and appendages on the geometry and consequently on heat exchange.
In this paragraph I do not make an approximation of the combination the sphere and the cone(beak). The effect, V_{sh}, is illustrated in Figure 3.1 below.

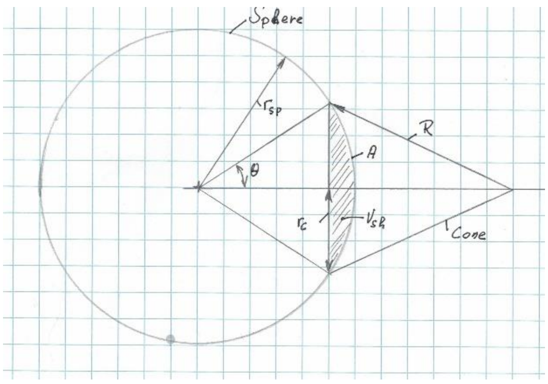


Figure 3.1 The Sphere as a primal body and the Cone(beak)

Neglecting the effect, we would have counted the shade area representing the volume V_{sh} twice. The cap area A would also be included in the area of the sphere.
In this paragraph we take the effect into account and compare the result obtained when neglecting the effect.
To find out about the area of the spherical cap A, I use the solid angle Ω, www.en.wikipedia.org .

For a sphere with radius unity

$\Omega = 2\pi(1 - \cos\theta)$.

Then,

$A = \Omega r_{sp}^2 = 2\pi(1 - \cos\theta) r_{sp}^2 = 2\pi\left(r_{sp}^2 - r_{sp}\sqrt{r_{sp}^2 - r_c^2}\right)$.

The segment of the sphere with solid angle Ω has a volume of

$\frac{1}{3}\Omega r_{sp}^3$.

For the volume of the cone with solid angle Ω and base radius r_c

$V_\Omega = \frac{1}{3}\pi r_c^2 \sqrt{r_{sp}^2 - r_c^2}$.

Finally, for the volume V_{sh}, the spherical cap

$$V_{sh} = \frac{1}{3}\Omega r_{sp}^3 - \frac{1}{3}\pi r_c^2\sqrt{r_{sp}^2 - r_c^2} = \frac{2\pi}{3}\left(1 - \frac{\sqrt{r_{sp}^2 - r_c^2}}{r_{sp}}\right) r_{sp}^3 - \frac{1}{3}\pi r_c^2 \sqrt{r_{sp}^2 - r_c^2} =$$

$$= \frac{2\pi}{3}\left(r_{sp}^3 - r_{sp}^2\sqrt{r_{sp}^2 - r_c^2}\right) - \frac{1}{3}\pi r_c^2\sqrt{r_{sp}^2 - r_c^2} = \frac{\pi}{3}\left[2r_{sp}^3 - \left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right].$$

Note: the ratio $\frac{V_{sh}}{V_{sp}} = \frac{1}{2}\left[1 - \left(1 + \frac{1}{2}\left(\frac{r_c}{r_{sp}}\right)^2\right)\sqrt{1 - \left(\frac{r_c}{r_{sp}}\right)^2}\right]$.

Then, with $\frac{r_c}{r_{sp}} \ll 1 \rightarrow \frac{V_{sh}}{V_{sp}} = \frac{1}{4}\left(\frac{r_c}{r_{sp}}\right)^2$.

For the sphere and the beak, the base(B) configuration for the ratio of the surface and the volume is

$$ratio_B = \frac{4\pi r_{sp}^2 + \pi r_c R - 2\pi\left(r_{sp}^2 - r_{sp}\sqrt{r_{sp}^2 - r_c^2}\right)}{\frac{4}{3}\pi r_{sp}^3 + \frac{\pi}{3} r_c^2\sqrt{R^2 - r_c^2} - \frac{\pi}{3}\left[2r_{sp}^3 - \left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right]} = 3\frac{2r_{sp}^2 + r_c R + 2r_{sp}\sqrt{r_{sp}^2 - r_c^2}}{2r_{sp}^3 + r_c^2\sqrt{R^2 - r_c^2} + \left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}} \equiv \frac{S_B}{V_B}.$$

We make $ratio_B$ dimensionless with

$\frac{r_c}{r_{sp}} = x$, $\frac{R}{r_{sp}} = y$, $\frac{S_B}{r_{sp}^2} = S$, and $\frac{V_B}{r_{sp}^3} = V$,

$$F(x, y) = r_{sp} \cdot ratio_B = 3\frac{2 + xy + 2\sqrt{1 - x^2}}{2 + x^2\sqrt{y^2 - x^2} + (2 + x^2)\sqrt{1 - x^2}} \equiv \frac{S}{B}.$$

To find out about the (local) stationary points, we need to calculate the derivatives, the zeros, and the Hessian matrix. Some calculus and a lot of algebra. Since $F(x, y)$ is a smooth function, I will use some plots of $F(x, y)$ to find out about the effect of the geometry on heat exchange with the environment.

In the Figure below a plot of $F(x, y)$ is presented:

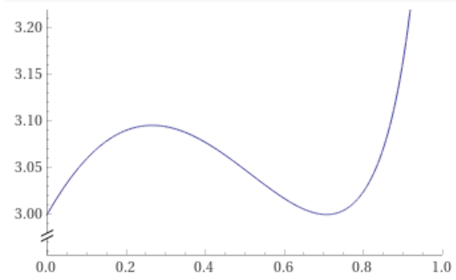


Figure 3.2 F(x,y) as a function of x with y=1

In Figure 3.2 the maximum value of heat exchange is found for $x \to 1$ the radius of the cone equal to the radius of the sphere and $F(1,1) = 4.5$. Another is found at $(x \cong 0.23, F(,x,y) = 3.10)$. A minimum is found at $(x \cong 0.76, F(x,y) \cong 3.01)$. So, in this case increasing x makes sense for a configuration with $x < 0.23$, and for $x > 0.76$. What about a configuration with $0.23 < x < 0.76$? What will the adaptation look like? Well, with other things equal, adaptation means reducing the dimensionless radius of the cone. However, in the real-world other things are not equal..

Note: $\frac{R}{r_{sp}} = \mathrm{y}$, the dimensionless slant height, is treated as a parameter. With the cone: $x < y$.

So, let us make a plot of $F(x,y)$ with $y = \frac{1}{2}$:

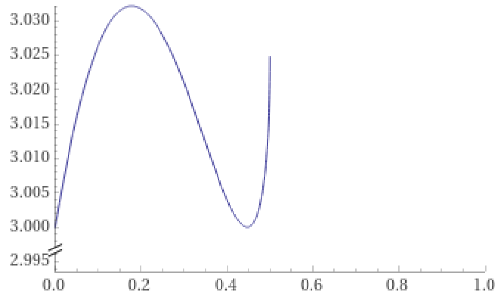


Figure 3.3 F(x,y) as a function of x with y=0.5

In Figure 3.3, the same pattern is observed as in Figure 3.2. Note the difference in the vertical scale. The maximum heat exchange is obtained at $(x \cong 0.19, F(x,y) \cong 3.032)$ and at $(x \cong 0.5, F(0.5,0.5) \cong 3.025)$. So, in this case increasing x makes sense for a configuration with $x < 0.19$, and for $x > 0.76$. What about a configuration with $0.19 < x < 0.46$? What will the adaptation look like? Again, adaptation means reducing the radius of the cone

Another plot of $F(x,y)$ with $y = 2$:

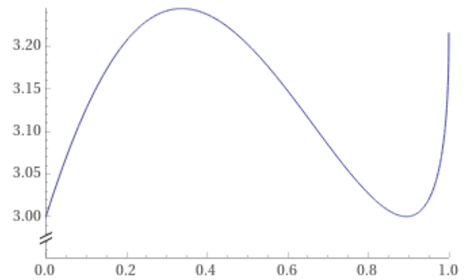


Figure 3.4 F(x,y) as a function of x with y=2

Again, the same pattern is observed in Figure 3.4. A maximum at $x \cong 0.35$ and at $x \cong 1$. A minimum is found at $x \cong 0.9$.

Next, I will plot $F(x,y)$ as a function of y with x as a parameter.

Let us plot $F(x, y)$ with $x = 0.25$. I start with a detail for $0.25 \leq y \leq 0.3$.

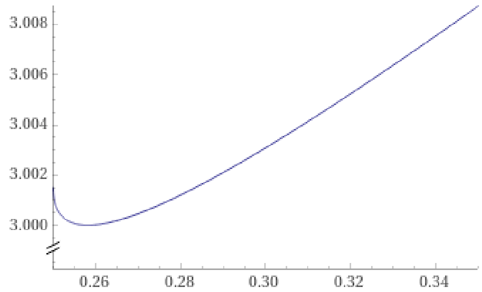


Figure 3.5 F(x,y) as a function of y with x=0.25

Comparing Figure 3.5 and Figure 3.6 it is clear why we illustrate the detail.
Now, I extend the range of y :

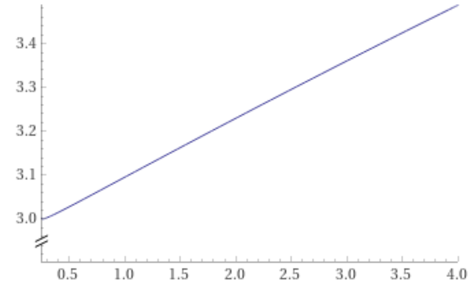


Figure 3.6 F(x,y) as a function of y with x=0.25

Figure 3.6 illustrates, after a small dip - Figure 3,5 - an increase of $F(x, y)$ from $x = 0.255$.
The following plot is with $x = 0.5$

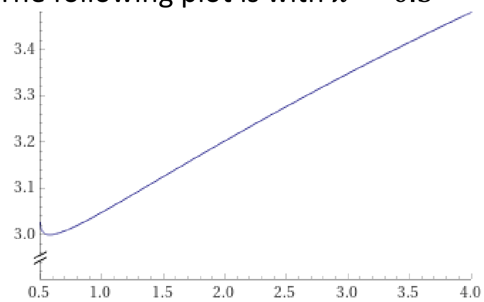


Figure 3.7 F(x,y) as a function of y with x=0.5

In Figure 3.7 the same pattern is illustrated as in Figure 3.5. The dip is clearly shown.
A plot of $F(x, y)$ with $x = 0.75$:

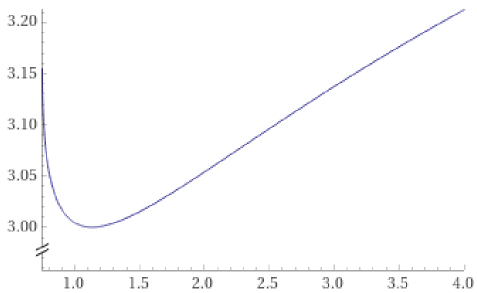


Figure 3.8 F(x,y) as a function of y with x=0.75

Here, for $x = 0.75$ the dip is emphasized.
So, what will happen with $x = 1$?

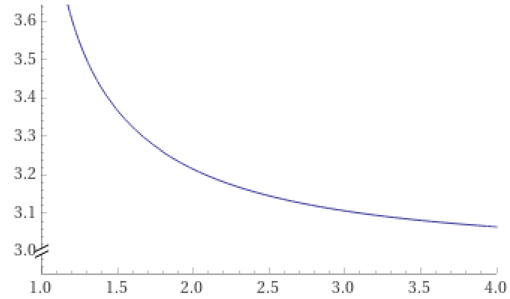


Figure 3.9 F(x,y) as a function of y with x=1

When $x = \frac{r_c}{r_{sp}} = 1$, increasing the slant height of the beak y decreases heat exchange with the environment. However, to have a functioning beak $y > x$.
With $x = 0.95$, $F(x, y)$ is decreasing and for $y > 2.5$, slightly increasing. Not shown here.

An interesting case left is optimising the volume of primal body and appendages for a given surface of the primal body and the appendages. Then, we have three independent variables in the case of the sphere and the cone. However, for a given surface one of the three independent variables can be expressed in the other two variables.

In the foregoing paragraphs I made an approximation of the effect on the geometry of the combination of primal body and the appendages. I assumed the effect to be small.
In the above analysis I included the effect on geometry of the combination of sphere and cone. In the note on page 18 the effect is small for with $\frac{r_c}{r_{sp}} \ll 1 \rightarrow \frac{V_{sh}}{V_{sp}} = \frac{1}{4}\left(\frac{r_c}{r_{sp}}\right)^2$.
For the area cut out of the sphere by the cone the base area of the cone can be used. For the cylinder as primal body with the cylindrical limbs we can use a similar approximation.
There is a possibility the sphere not to contribute to the heat exchange. In that case one could think the analysis of § 2.3 should be used. However, the sphere still plays a role in the metabolism and consequently in process of heat exchange. By excluding the surface of the sphere in the ratio of surface to volume, the effect can be analysed.

§ 3.4 The Sphere as the Primal Body and Pyramid(Beak)

In the Figure below the geometry to be analysed is illustrated.

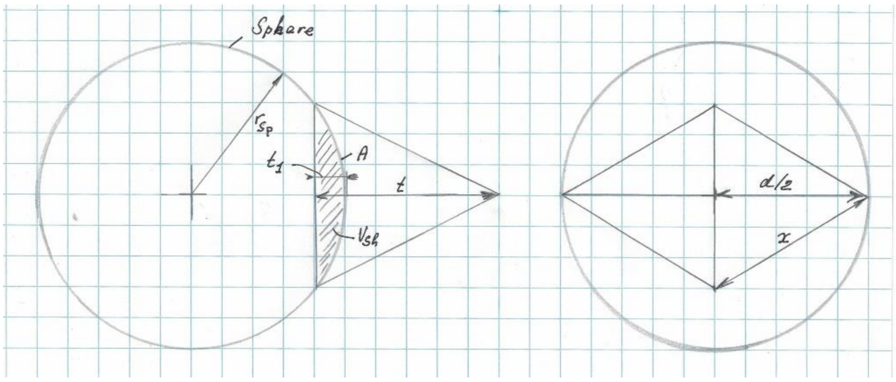


Figure 3.10 The Sphere as a primal body and the Rhombic Pyramid(Beak)

In Figure 3.10, the configuration is illustrated.
Investigating this configuration, the analysis is almost lost in detail.
The major issue here is to optimize heat exchange for the combination of primal body and appendages. So, when applying approximations I prefer the use of a cube as primal body for the analysis.

The surface of the pyramid, § 2.4 see Figures 2.6 and 2.7, without the area of the base of the pyramid

$S_p = \sqrt{t^2(h^2 + w^2) + \frac{1}{4}h^2w^2}$

With c the edge of the cube , the area of the cube $6c^2$reduced with the area of the base of the pyramid

$S_c = 6c^2 - \frac{1}{2}hw$.

Reminder, in the analysis of this paragraph the cube is treated to be a constant→ c is a constant number.

The volume of the cube and the pyramids is:

$V_{c+p} = c^3 + \frac{1}{3} \cdot \frac{1}{2} \cdot \ hw \cdot t.$

To investigate the configuration cube + pyramid, I set $c = w$, a constant.

The ratio surface to volume is:

$$ratio_{c+p} = \frac{6c^2 - \frac{1}{2}hw + \sqrt{t^2(h^2+w^2)+\frac{1}{4}h^2w^2}}{c^3+\frac{1}{3}\cdot\frac{1}{2}\cdot hw\cdot t} = \frac{6c^2 - \frac{1}{2}hc + \sqrt{t^2(h^2+c^2)+\frac{1}{4}h^2c^2}}{c^3+\frac{1}{3}\cdot\frac{1}{2}\cdot hc\cdot t}$$

We have: $ratio_{c+p} = ratio_{c+p}(h, t)$.

To obtain a stationary point determine $\frac{\partial ratio_{c+p}}{\partial h} = 0$ and $\frac{\partial ratio_{c+p}}{\partial t} = 0$.

For a point with both derivatives zero, we like to know whether we are dealing with a maximum, a minimum or a saddle point. To this end we need the Hessian matrix, the determinant of this matrix and its trace. To construct the matrix, we derive the second derivatives of $ratio_B$. There are four and the Hessian matrix is a 2×2 symmetric matrix. See Appendix.

We make $ratio_{c+p}$ dimensionless with c. The new variables are primed. In the following I drop the primes:

$$F(h, t) = c \cdot ratio_B = \frac{6 - \frac{h}{2} + \sqrt{t^2h^2+t^2+\frac{1}{4}h^2}}{1+\frac{1}{6}h\cdot t} .$$

As mentioned above, to investigate the possibility of stationary point(s), we can calculate the derivatives with respect to h and t. I will not do that.

Here, we present $F(h,t)$ as a function of $0 < h < 1$ with $t = 0.5, 1$, and 2.

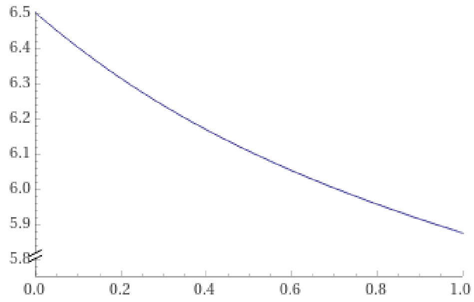


Figure 3.11 F(h,t) as a function of h with t=0.5

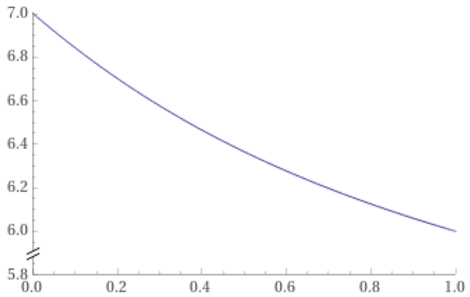


Figure 3.12 F(h,t)as a function of h with t=1

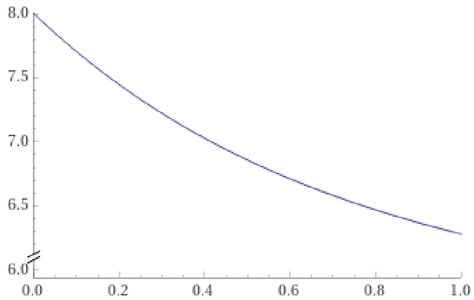


Figure 3.13 F(h,t) as a function of h with t=2

The above three Figures illustrate the effect of morphological change on heat exchange. Considering the scale of the above Figures, it appears a decrease of h and increase of t improves heat exchange with the environment.

§ 3.5 The Sphere as the Primal Body and the Cylinder as Beak.

For the analysis of the sphere and the cylinder I use some of the results of the sphere and the cone, see Figure 3.1 page 14:

- $A = 2\pi\left(r_{sp}^2 - r_{sp}\sqrt{r_{sp}^2 - r_c^2}\right)$,

where r_c is the radius of the cylinder.

- $V_{sh} = \frac{\pi}{3}\left[2r_{sp}^3 - \left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right]$.

The ratio surface to volume is:

$$ratio_B = \frac{4\pi r_{sp}^2 + 2\pi r_c L - 2\pi\left(r_{sp}^2 - r_{sp}\sqrt{r_{sp}^2 - r_c^2}\right) + \pi r_c^2}{\frac{4}{3}\pi r_{sp}^3 + \pi r_c^2 L - \frac{\pi}{3}\left[2r_{sp}^3 - \left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right]},$$

where L is the length of the cylinder.

$$ratio_B = \frac{2r_{sp}^2+2r_cL+2r_{sp}\sqrt{r_{sp}^2-r_c^2}+\pi r_c^2}{\frac{2}{3}r_{sp}^3+r_c^2L+\frac{1}{3}\left[(2r_{sp}^2+r_c^2)\sqrt{r_{sp}^2-r_c^2}\right]} \equiv \frac{S_B}{V_B}.$$

We make the preceding expression dimensionless with r_{sp}(kept constant)

$\frac{r_c}{r_{sp}} = x, \frac{L}{r_{sp}} = y, \frac{S_B}{r_{sp}^2} = S$, and $\frac{V_B}{r_{sp}^3} = V$.

Then,

$$F(x,y) = r_{sp} \cdot ratio_B = \frac{2+2xy+x^2+2\sqrt{1-x^2}}{\frac{2}{3}+x^2y+\frac{1}{3}(1+x^2)\sqrt{1-x^2}},$$

with $x \leq 1$.

We illustrate the effect of x and y on $F(x,y)$ by plotting $F(x,y)$, where I treat x and y as a parameter as done in the foregoing sections.

- A plot of $F(x,y)$ as a function of x with $y = 0.5$:

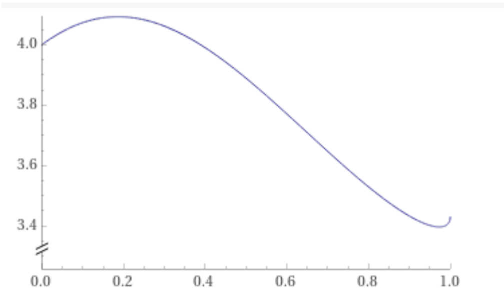


Figure 3.14 F(x,y) as a function of x with y=0.5

For $0 < x < 2.2$, with at $x = 0.2$ the maximum value of F, heat exchange can be enhanced by some 5%.

With $y = \frac{3}{4}$ and $y = 1$, similar plots are obtained as above leading to the same conclusion with respect to heat exchange.

With $y = 2$, heat exchange can be increased by 15% for $0 < x < 2.2$.

- Next we investigate a plot of $F(x,y)$ as a function of y with $x = 0.25$:

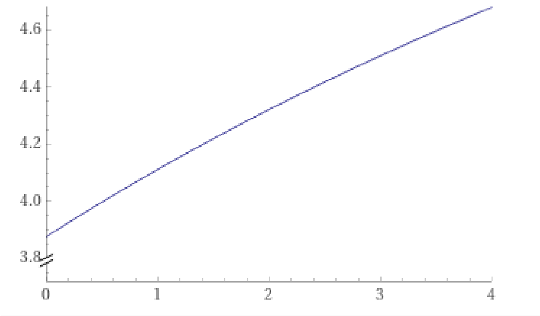


Figure 3.15 F(x,y) as a function of y with x=0.25

A plot with $x = 0.5$

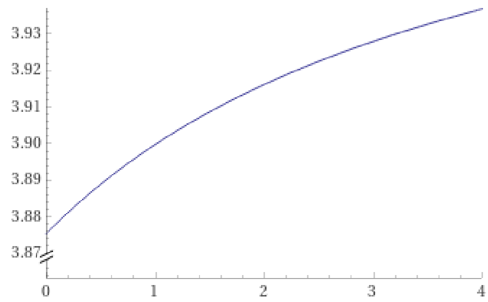


Figure 3.16 F(x,y) as a function of y with x=0.5

A plot with $x = 1$

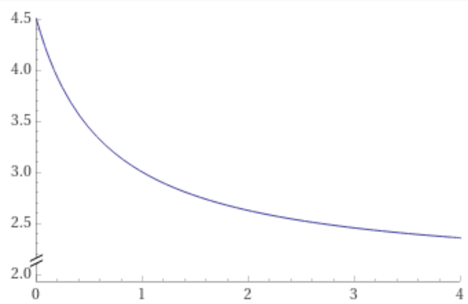


Figure 3.17 F(x,y) as a function of y with x=1

$x = 0.75$:

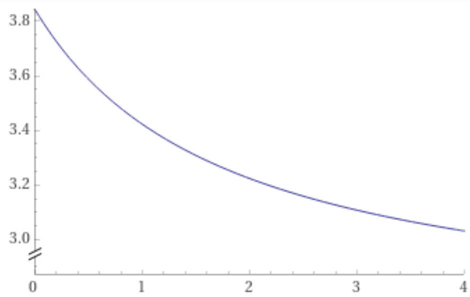


Figure 3.18 F(x,y) as a function of y with x=0.75

$x = 0.6$:

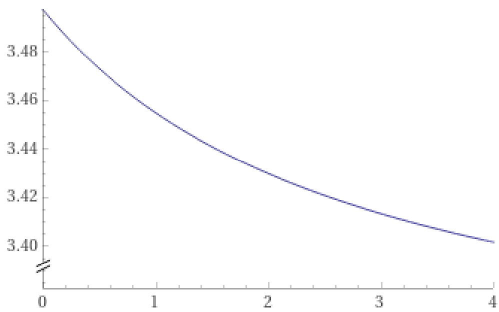


Figure 3.19 F(x,y) as a function of y with x=0.6

With $x = 0.55$, we have a similar plot as for $x = 0.6$ except for $F(0.55,0) = 3.86$.

$x = 0.52$:

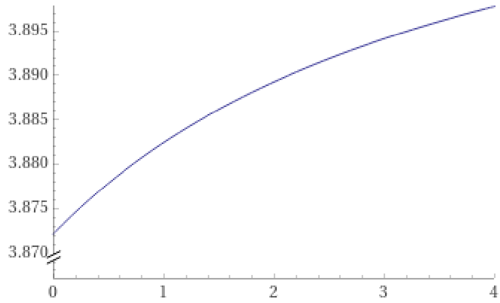


Figure 3.20 F(x,y) as a function of y with x=0.52

With $x = 0.52$, we have a similar plot as for $x = 0.6$ except for $F(0.52,0) = 3.867$.
Calculating $\frac{\partial F}{\partial y}$, and x to be a parameter, we find with

$F(x,y) = \frac{2+2xy+x^2+2\sqrt{1-x^2}}{\frac{2}{3}+x^2y+\frac{1}{3}(1+x^2)\sqrt{1-x^2}}$, and $\frac{\partial F}{\partial y} = 0 \Longrightarrow x = 0.517$.

We assume the maximum radius of the beak to be equal to the radius of the sphere.
So together with $x = 0.5$, and in the range $0 < y < 4$ maximum heat exchange is obtained at $(0.5,4)$. However, in the range of $0 < y < 4$, together with $x = 0.5$, the increase of heat exchange is of the order of 1%.
A plot of $F(x,y)$ together with $y = 4$, and in the range $0 < x < 1$ shows a growth in heat exchange of about 20% for x from 0 to 0.22. This is illustrated by Figure 3.21:

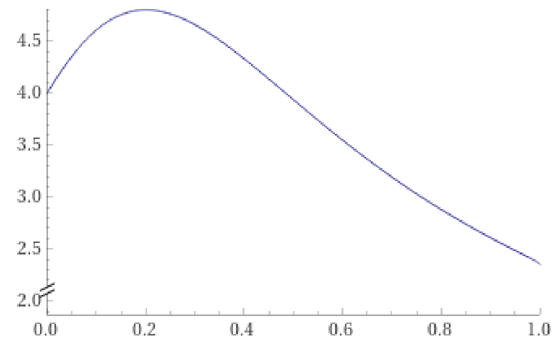


Figure 3.21 F(x,y) as a function of x with y=4

§ 4 Optimizing Primal Body and the Appendages

In the preceding paragraphs the geometry of the primal body is constant, and the appendages are optimized.
In the real world the total geometry of the vertebrate will adapt. In this section we will investigate this adaptation.
There are two approaches:
- Optimizing the surface for a given volume,
- Optimizing the volume for a given surface.

§ 4.1 Optimize the Sphere and the Four Limbs.

I will use the same notation as used in the preceding paragraphs.

The surface:

$S = 4\pi r_{sh}^2 + 8\pi r_{cl} L.$

The volume:

$V = \frac{4}{3}\pi r_{sh}^3 + 4\pi r_{cl}^2 L.$

We use two approaches

- Optimize the surface for a given volume,
- Optimize the volume for a given surface.

§ 4.1.1 Optimize the Surface for a given Volume V.

We have three variables: r_{sh}, r_{cl} and L.

For a given volume V we express L in r_{sh}, r_{cl} and V:

$L = \frac{V - \frac{4}{3}\pi r_{sh}^3}{4\pi r_{cl}^2}.$

Then,

$S = 4\pi r_{sh}^2 + 8\pi r_{cl} \frac{V - \frac{4}{3}\pi r_{sh}^3}{4\pi r_{cl}^2} = 4\pi r_{sh}^2 + \frac{2V}{r_{cl}} - \frac{8}{3}\pi \frac{r_{sh}^3}{r_{cl}}.$

Next, we make the preceding expression dimensionless with $V^{2/3}$:

$\frac{S}{V^{\frac{2}{3}}} = \frac{4\pi r_{sh}^2}{V^{2/3}} + \frac{2V}{V^{2/3} r_{cl}} - \frac{8}{3}\pi \frac{r_{sh}^3}{V^{\frac{2}{3}} r_{cl}} = 4\pi \left(\frac{r_{sh}}{V^{\frac{1}{3}}}\right)^2 + 2\frac{V^{1/3}}{r_{cl}} - \frac{8}{3}\pi \frac{r_{sh}^3}{V} \frac{V^{\frac{1}{3}}}{r_{cl}}.$

Substitute

$\frac{r_{sh}}{V^{\frac{1}{3}}} = x$, $\frac{r_{cl}}{V^{\frac{1}{3}}} = y$, and $\frac{S}{V^{\frac{2}{3}}} = S.$

The result

$S = 4\pi x^2 + \frac{2}{y} - \frac{8}{3}\pi \frac{x^3}{y}.$

In addition a dimensionless L:

$L \equiv \frac{L}{V^{\frac{1}{3}}} = \frac{1 - \frac{\frac{4}{3}\pi r_{sh}^3}{V}}{\frac{4\pi r_{cl}^2}{V^{\frac{2}{3}}}} = \frac{1 - \frac{4}{3}\pi x^3}{4\pi y^2},$

with $L > 0 \Longrightarrow \frac{4}{3}\pi x^3 < 1.$

In § 3.1, another, biological, limit is presented $\frac{x}{y} > \sqrt{2}$.

Consequently,

$y\sqrt{2} < x < \left(\frac{3}{4\pi}\right)^{\frac{1}{3}} (= 0.62).$

So, $y < 0.44.$

Stationary points:

$\frac{\partial S}{\partial x} = 8\pi x - 8\pi \frac{x^2}{y},$

and

$\frac{\partial S}{\partial y} = -\frac{2}{y^2} + \frac{8}{3}\pi \frac{x^3}{y^2}.$

As explained in the Appendix 1, we need the second derivatives in order to construct the

Hessian matrix:

- $\frac{\partial^2 S}{\partial x^2} = 8\pi - 16\pi \frac{x}{y}$,
- $\frac{\partial^2 S}{\partial y^2} = \frac{4}{y^3} - \frac{16}{3}\pi(\frac{x}{y})^3$,
- $\frac{\partial^2 S}{\partial x \partial y} = \frac{\partial^2 S}{\partial y \partial x} = 8\pi(\frac{x}{y})^2$.

We have all the ingredients to decide about the stationary points:

- $\frac{\partial S}{\partial x} = 8\pi x - 8\pi \frac{x^2}{y} = 0 = x\left(1 - \frac{x}{y}\right) \Longrightarrow x = 0, x = y$.
- $\frac{\partial S}{\partial y} = -\frac{2}{y^2} + \frac{8}{3}\pi \frac{x^3}{y^2} = 0 = \frac{1}{y^2}\left(\frac{4}{3}\pi x^3 - 1\right) \Longrightarrow y \to \infty, x = (\frac{3}{4\pi})^{1/3}$.

Stationary point: $(\left[\frac{3}{4\pi}\right]^{\frac{1}{3}}, \left[\frac{3}{4\pi}\right]^{\frac{1}{3}})$.

Notice: $x < (\frac{3}{4\pi})^{1/3}$.

The elements of the determinant of the Hessian matrix:

- $\frac{\partial^2 S}{\partial x^2} = 8\pi - 16\pi \frac{x}{y} = -8\pi$,
- $\frac{\partial^2 S}{\partial y^2} = \frac{4}{y^3} - \frac{16}{3}\pi\left(\frac{x}{y}\right)^3 = \frac{16\pi}{3} - \frac{16\pi}{3} = 0$,
- $\frac{\partial^2 S}{\partial x \partial y} = \frac{\partial^2 S}{\partial y \partial x} = 8\pi(\frac{x}{y})^2 = 8\pi$.

Hence, the determinant of the matrix is negative and the stationary point is a saddle point.

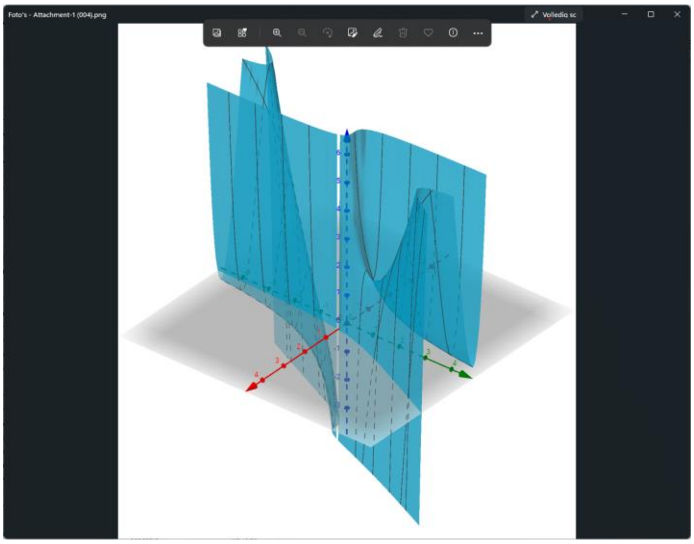


Figure 4.1 S(x,y) 3-D representation for a given volume with GeoGebra

The red arrow pointing downwards represent the positive x-axis, the green arrow, pointing to the right downwards the positive y-axis. For further analysis we investigate cross sections of $S(x, y)$.

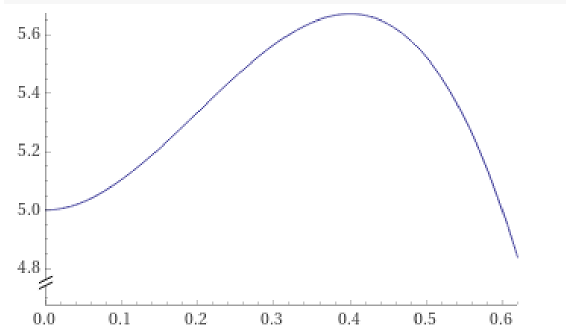


Figure 4.2 S(x,y) as a function of x with y=0.4

With $y = 0.4$, the maximum value of $S(x, y) = 5.66$, is obtained at $x = 0.4$.
For the combination $(x, y) = (0.4,0.4)$, we find $L = 0.092$.
Well, here we take note of not including the biological limit $x > \sqrt{2}$. Consequently, the possible maximum value of $S(x, y)$ at $x = 0.4 \cdot \sqrt{2} \cong 0.56$: $S(0.56,0.4) = 5.26$.
This results into $L = 0.13$.
Hence, the diameter of the limbs, 0.4, is larger than the length of the limbs, 0.13.
Another cross section with $y = 0.2$:

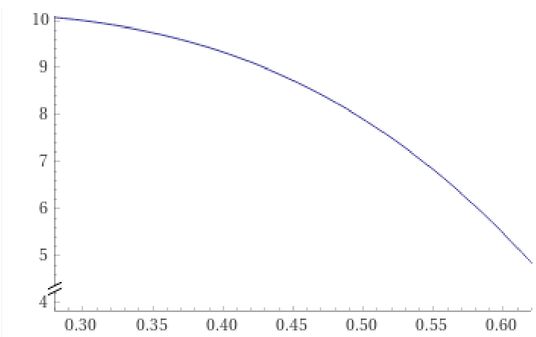


Figure 4.3 S(x,y) as a function of x with y=0.2

In Figure 4.3 it is illustrated $S = 10.1$ to be a maximum for the smallest possible value of $x = 2.8$.
In the Figure below the length L is plotted as a function of x and $y = 0.2$.

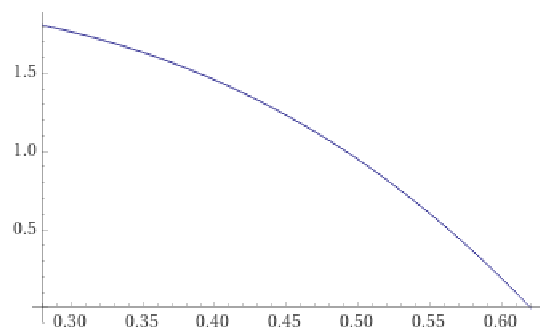


Figure 4.4 L(x,y) as a function of x with y=0.2

The maximum value of $L = 1.8$ is obtained at the lowest possible value of $x = 0.28$.
It appears the heat exchange to be increased for decreasing the dimensionless radii x and y, and increasing the dimensionless length of the four limbs. Obviously, up to a point. Think about the function of the limbs and its strength.

§ 4.1.2 Optimize the Volume for a given surface S.

We have three variables: r_{sh}, r_{cl} and L.
For a given volume S we express L in r_{sh}, r_{cl} and S:

$$S = 4\pi r_{sh}^2 + 8\pi r_{cl} L \Longrightarrow L = \frac{S-4\pi r_{sh}^2}{8\pi r_{cl}} .$$

The volume:

$$V = \frac{4}{3}\pi r_{sh}^3 + 4\pi r_{cl}^2 L = \frac{4}{3}\pi r_{sh}^3 + 4\pi r_{cl}^2 \frac{S-4\pi r_{sh}^2}{8\pi r_{cl}} = \frac{4}{3}\pi r_{sh}^3 + \frac{r_{cl}S}{2} - 2\pi r_{cl} r_{sh}^2.$$

The preceding expression is made dimensionless with $S^{3/2}$:

$$\frac{V}{S^{3/2}} = \frac{4}{3}\pi \frac{r_{sh}^3}{S^{3/2}} + \frac{r_{cl}}{2S^{1/2}} - 2\pi \frac{r_{cl} r_{sh}^2}{S^{3/2}} .$$

With

$x = \frac{r_{sh}}{S^{1/2}}$, $y = \frac{r_{cl}}{S^{1/2}}$, and $V = \frac{V}{S^{3/2}}$

$$V = \frac{4}{3}\pi x^3 + \frac{1}{2}y - 2\pi y x^2.$$

The dimensionless L:

$$L = \frac{L}{S^{1/2}} = \frac{1-4\pi x^2}{8\pi y}.$$

From the preceding expression we learn:

$$1 - 4\pi x^2 > 0.$$

With the biological constraint $x > y\sqrt{2}$, we find the range for x :

$$y\sqrt{2} < x < \frac{1}{2\sqrt{\pi}} (= 0.28).$$

What about the stationary points?

$$\frac{\partial V}{\partial x} = 4\pi x^2 - 4\pi xy .$$

$$\frac{\partial V}{\partial x} = 0 = 4\pi x^2 - 4\pi xy .$$

Then,

$$x(x-y) = 0 \Longrightarrow x = 0, x = y.$$

$$\frac{\partial V}{\partial y} = \frac{1}{2} - 2\pi x^2 .$$

$$\frac{\partial V}{\partial y} = 0 = \frac{1}{2} - 2\pi x^2.$$

Then

$$x = \frac{1}{2\sqrt{\pi}} .$$

The elements of the determinant of the Hessian matrix:

- $\frac{\partial^2 V}{\partial x^2} = 8\pi x - 4\pi y$,
- $\frac{\partial^2 V}{\partial y^2} = 0$,
- $\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 S}{\partial y \partial x} = -4\pi x$.

The trace of the matrix is irrelevant, since the determinant is negative: a saddle point.

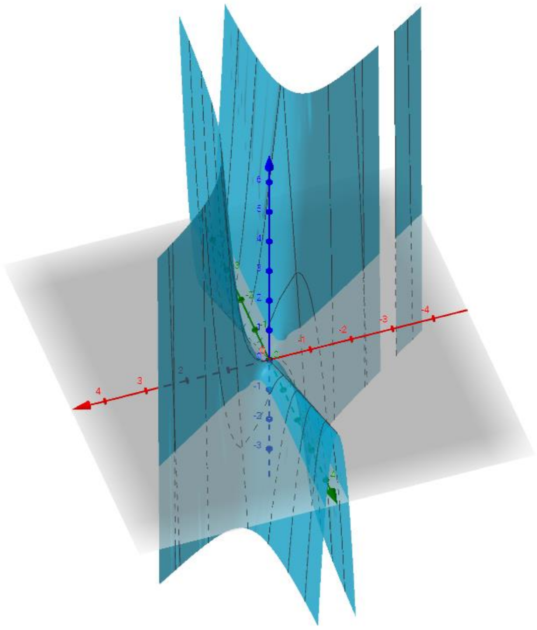


Figure 4.5 V(x,y) a 3-D representation for a given surface

Next, I will present some cross sections.
It is about optimizing to improve heat exchange.
Hence, for a given surface we look for the lowest possible volume.
For $y = 0.15$:

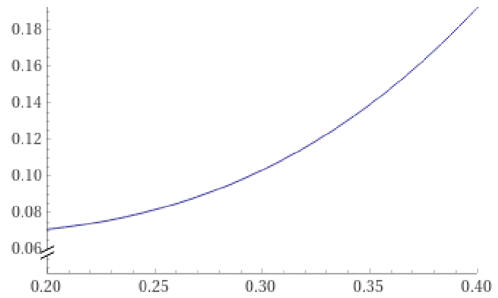


Figure 4.6 V(x,y) as a function of x with y=0.15

With the biological constraint $x > y\sqrt{2}$, we obtained the range for x :
$y\sqrt{2} < x < \frac{1}{2\sqrt{\pi}} (= 0.28)$.
The lowest possible value of $V(x, y)$ is at $x = 0.21$. Then, $V = 0.072$.
For $y = 0.1$:

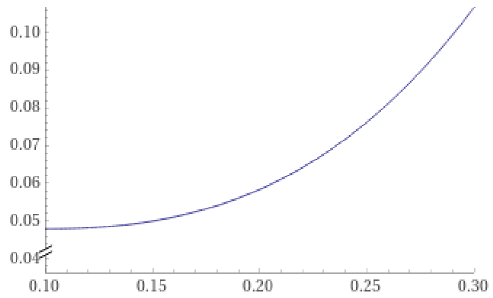


Figure 4.7 V(x,y) as function of x with y=0.1

The lowest possible value of $V(x, y)$ is at $x = 0.14$. Then, $V = 0.049$.
For $y = 0.05$:

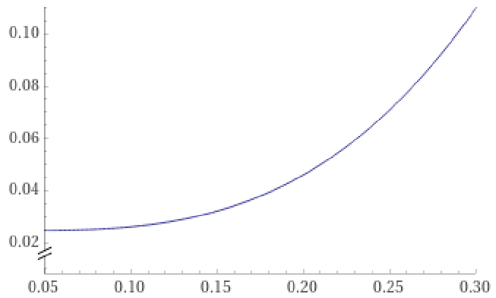


Figure 4.8 V(x,y) as a function of x with y=0.05

The lowest possible value of $V(x, y)$ is at $x = 0.075$. Then, $V = 0.03$.

§ 4.2 Optimize the Cylinder and the four Limbs

§ 4.2.1 Optimize the Surface for a given Volume V

Now we have to deal with four variables, see § 3.2
- r the radius of the cylinder as Primal Body,
- L the length of the cylinder as Primal Body,
- r_{cl} the radius of the cylinder as Limb,
- l the length of the cylinder as Limb.

The surface S

$S = 2\pi rL + 2\pi r^2 + 8\pi r_{cl}l,$

the volume V

$V = \pi r^2 L + 4\pi r_{cl}^2 l.$

We can reduce the four independent variables by expressing one in V and the other three variables.

Then, to use the analytical approach, we need to make cross section for three independent variables instead of two. In addition, to present realistic plots use needs to be made of the so-called biological constraints as mentioned in § 3.2.

§ 4.2.2 Optimize the Volume for a given Surface.

The same remarks as made in § 4.2.1 apply to this case.

§ 4.3 Optimise the Sphere and the Cone

§ 4.3.1 Optimize the Surface S for a given Volume V

In this case we have three independent variables, see § 3.3.

The surface S

$$S = 4\pi r_{sp}^2 + \pi r_c R - 2\pi\left(r_{sp}^2 - r_{sp}\sqrt{r_{sp}^2 - r_c^2}\right) = 2\pi r_{sp}^2 + \pi r_c R + 2\pi r_{sp}\sqrt{r_{sp}^2 - r_c^2},$$

the volume V

$$V = \frac{4}{3}\pi r_{sp}^3 + \frac{\pi}{3} r_c^2\sqrt{R^2 - r_c^2} - \frac{\pi}{3}\left[2r_{sp}^3 - \left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right] =$$

$$= \frac{2}{3}\pi r_{sp}^3 + \frac{\pi}{3} r_c^2\sqrt{R^2 - r_c^2} + \frac{\pi}{3}\left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\ .$$

Three independent variables:

- r_{sp} the radius of the sphere,
- r_c the radius of the cone,
- R the slant height of the cone.

We can reduce the three independent variables by expressing one in V and the other two variables.

To illustrate the effect of optimizing, set $r_{sp} = r_c$.

So,

$$S = 2\pi r_{sp}^2 + \pi r_{sp} R,$$

and

$$V = \frac{2}{3}\pi r_{sp}^3 + \frac{\pi}{3} r_{sp}^2\sqrt{R^2 - r_{sp}^2}\ .$$

Next we express R into r_{sp} and V:

$$R = [\left(\frac{3V}{\pi r_{sp}^2}\right)^2 - \frac{12V}{\pi r_{sp}} + 5r_{sp}^2]^{1/2}.$$

For the surface to be optimized, we have:

$$S = 2\pi r_{sp}^2 + \pi r_{sp}[\left(\frac{3V}{\pi r_{sp}^2}\right)^2 - \frac{12V}{\pi r_{sp}} + 5r_{sp}^2]^{1/2} = 2\pi r_{sp}^2 + [\left(\frac{3V}{r_{sp}}\right)^2 - 12\pi V r_{sp} + 5\pi^2 r_{sp}^4]^{1/2}.$$

The next step is to make the preceding expression of S dimensionless and denote

$\frac{S}{V^{2/3}} = y$, and $\frac{r_{sp}}{V^{1/3}} = x$,

$$y = 2\pi x^2 + [\frac{9}{x^2} - 12\pi x + 5\pi^2 x^4]^{1/2}.$$

The dimensionless slant height $\frac{R}{V^{1/3}} = z$

$$z = [\frac{9}{\pi^2 x^4} - \frac{12}{\pi x} + 5x^2]^{1/2}.$$

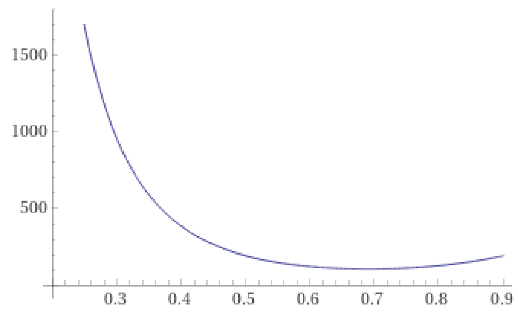


Figure 4.9 S(=y) as a function of x

Decreasing x, Figure 4.9, left of the minimum value of y at $x = 0.692$, creates efficient heat exchange .

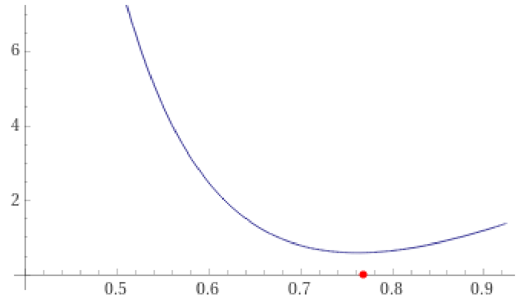


Figure 4.10 R(=z) as a function of x

The slant height $R(= z)$, Figure 4.10, increases as a function of x left of the minimum value of z at $x \cong 0.765$.

§ 4.3.2 Optimize the Volume V for a given Surface S

The surface S

$$S = 2\pi r_{sp}^2 + \pi r_c R + 2\pi r_{sp}\sqrt{r_{sp}^2 - r_c^2},$$

the volume V

$$V = \frac{2}{3}\pi r_{sp}^3 + \frac{\pi}{3} r_c^2 \sqrt{R^2 - r_c^2} + \frac{\pi}{3}\left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2} .$$

Three independent variables:

- r_{sp} the radius of the sphere,
- r_c the radius of the cone,
- R the slant height of the cone.

We can reduce the three independent variables by expressing one in S and the other two variables.

To illustrate the effect of optimizing, set $r_{sp} = r_c$.

So,

$$S = 2\pi r_{sp}^2 + \pi r_{sp} R,$$

and

$$V = \frac{2}{3}\pi r_{sp}^3 + \frac{\pi}{3} r_{sp}^2 \sqrt{R^2 - r_{sp}^2} .$$

Next we express R into r_{sp} and S:

$$R = \frac{S}{\pi r_{sp}} - 2r_{sp}.$$

Plug the preceding expression into

$$V = \frac{2}{3}\pi r_{sp}^3 + \frac{\pi}{3} r_{sp}^2 \sqrt{R^2 - r_{sp}^2} \Rightarrow V = \frac{2}{3}\pi r_{sp}^3 + \frac{\pi}{3} r_{sp}^2 \sqrt{\left(\frac{S}{\pi r_{sp}} - 2r_{sp}\right)^2 - r_{sp}^2}.$$

The next step is to make the preceding expression of V dimensionless and denote $\frac{V}{S^{3/2}} = y$, and $\frac{r_{sp}}{S^{1/2}} = x$,

$$y = \frac{2}{3}\pi x^3 + \frac{\pi}{3} x^2 \sqrt{\left(\frac{1}{\pi x} - 2x\right)^2 - x^2} = \frac{2}{3}\pi x^3 + \frac{\pi}{3} x^2 \sqrt{\left(\frac{1}{\pi x}\right)^2 - \frac{4}{\pi} + 3x^2} .$$

The dimensionless slant height $\frac{R}{S^{1/2}} = z$:

$$z = \frac{1}{\pi \mathrm{x}} - 2\mathrm{x} .$$

A plot of $y = \frac{2}{3}\pi x^3 + \frac{\pi}{3}x^2\sqrt{(\frac{1}{\pi x})^2 - \frac{4}{\pi} + 3x^2}$ is shown in Figure 4.11

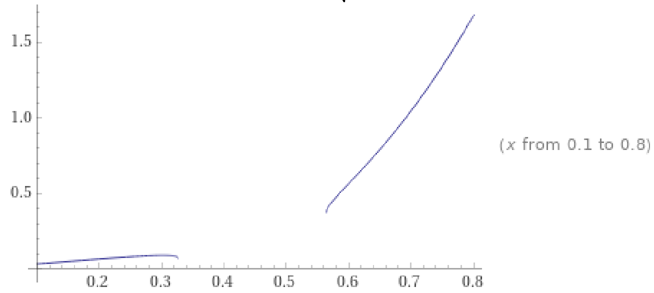


Figure 4.11 V(=y) as a function of x

The above figure shows a gap between $x = \frac{1}{\sqrt{3\pi}}$, and $x = \frac{1}{\sqrt{\pi}}$ since of the following expression

$\frac{\pi}{3}x^2\sqrt{(\frac{1}{\pi x})^2 - \frac{4}{\pi} + 3x^2} \equiv \frac{x}{3}\sqrt{1 - 4\pi x^2 + 3\pi^2 x^4} \Longrightarrow 1 - 4\pi x^2 + 3\pi^2 x^4 \leq 0$ for this range of x.

Furthermore, the above Figure 4.11, illustrates with $x > \frac{1}{\sqrt{\pi}}$, decreasing x creates efficient heat exchange. However, the slant height is negative for $x > \frac{1}{\sqrt{\pi}}$.

$z = \frac{1}{\pi \mathrm{x}} - 2\mathrm{x} \Longrightarrow \mathrm{z} = 0 = \frac{1}{\pi \mathrm{x}} - 2\mathrm{x} \Longrightarrow \mathrm{x} = \frac{1}{\sqrt{2\pi}}$.

From the above analysis we conclude to improve heat exchange with the environment is increasing S for a given volume.

§ 4.4 Optimize the Sphere and the Pyramid

§ 4.4.1 Optimize the Surface S for a given Volume V

The major issue is to optimize heat exchange for the combination of primal body and appendages. So, when applying approximations, I prefer the use of a cube as primal body for the analysis. The pyramid is investigated in § 3.4. There, instead of a sphere a cube is used.

The surface of the cube and pyramid:

$$S = 6c^2 - \frac{1}{2}hw + \sqrt{t^2(h^2 + w^2) + \frac{1}{4}h^2w^2}$$

the volume of the cube and sphere

$$V_{c+p} = c^3 + \frac{1}{3}\cdot\frac{1}{2}\cdot\ hw \cdot t.$$

where c is the edge of the cube, and I reduced the surface of the cube with the area cut out of the cube by the basis of the pyramid. I set the width of the beak(pyramid) $w = c$, se Figures 2.6 and 2.7.

So,

$$S = 6c^2 - \frac{1}{2}hc + \sqrt{t^2(h^2 + c^2) + \frac{1}{4}h^2c^2},$$

$S = S(c, t, h)$, and $V_{c+p} = V(c, t, h)$, both functions of three variables.

Since, we analyse for a given V, we express t in V, c and h :

$$t = \frac{6(V - c^3)}{h \cdot c}.$$

Plug the latter expression into S:

$$S = 6 \cdot c^2 - \frac{1}{2} h \cdot c + \sqrt{[\frac{6(V-c^3)}{h \cdot c}]^2 + \frac{1}{4}(h \cdot c)^2}.$$

Now $S = S(c, h)$ is dependent on two independent variables.
In Appendix 1, we presented the procedure to find the stationary points of a function dependent on two independent variables.
We can do a bit more to simplify the function $S = S(c, h)$, since the assumption h increases with increasing c and consequently w, the width of the beak, is realistic. Here I assume a linear relation
$h = a \cdot c$,
with $0 < a \leq 1$.
We make the expression for S dimensionless
$S' \equiv F = \frac{S}{V^{\frac{2}{3}}}, c' = \frac{c}{V^{1/3}}$, and $h' = \frac{h}{V^{\frac{1}{3}}}$.
Plug the preceding expressions in S, after dropping the primes:

$$S = (6 - \frac{1}{2} a) c^2 + \sqrt{\frac{1}{c} [\frac{6(1-c^3)}{a}]^2 + \frac{1}{4} a^2 \cdot c^4}.$$

To find out about the maximum value of S, we could calculate $\frac{\partial S}{\partial c}$ and $\frac{\partial^2 S}{\partial c^2}$.
However, I use the graphical approach. By making the variables dimensionless using V, makes the graphical approach feasible since $h = a \cdot c = O(1)$, and $c = O(1)$. A few graphs are plotted.
A graph of S with $a = \frac{1}{4}$:

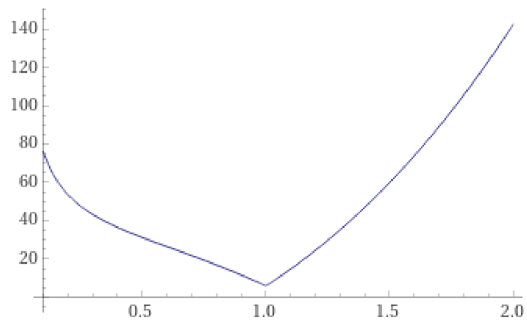


Figure 4.12 S(c) as a function of c with a=1/4

The above graph shows for $c < 1$, decreasing the size of the cube leads to an increasing surface and consequently improvement of the heat exchange. For $c > 1$, increasing the size of the cube leads to an increasing surface and consequently improvement of the heat exchange.

A graph of S with $a = \frac{1}{2}$:

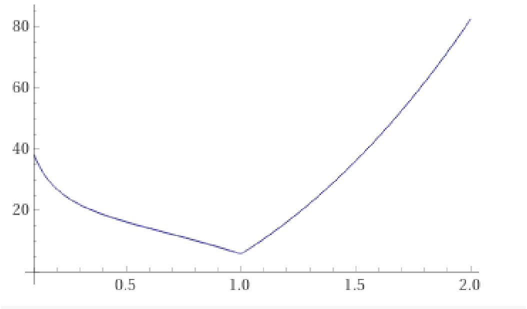


Figure 4.13 S(c) as a function of c with a=1/2

This graph shows the same pattern as presented in Figure 4.12.

A graph of S with $a = 1$:

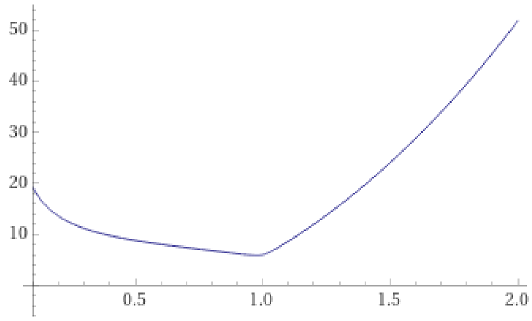


Figure 4.14 S(c) as a function of c up to 2, with a=1

This graph shows the same pattern as presented in Figure 4.12.

§ 4.4.2 Optimize the Volume V for a given Surface S

The surface

$$S = 6c^2 - \frac{1}{2}hw + \sqrt{t^2(h^2 + w^2) + \frac{1}{4}h^2w^2}\,.$$

The volume of the cube and sphere

$$V_{c+p} = c^3 + \frac{1}{3} \cdot \frac{1}{2} \cdot\ hw \cdot t.$$

where c is the edge of the cube, and I reduced the surface of the cube with the area cut out of the cube by the basis of the pyramid. Set the width of the beak(pyramid) $w = c$, see Figures 2.6 and 2.7.

So,

$$S = 6c^2 - \frac{1}{2}hc + \sqrt{t^2(h^2 + c^2) + \frac{1}{4}h^2c^2}\,.$$

Since, we analyse for a given S, we express t in S, c and h :

$$t = \sqrt{\frac{(S - 6c^2 + \frac{1}{2}hc)^2}{h^2 + c^2} - \frac{1}{4}h^2c^2}\,.$$

The volume of the cube and sphere

$$V_{c+p} = c^3 + \frac{1}{3}\cdot\frac{1}{2}\cdot\ hc\cdot\sqrt{\frac{(S-6c^2+\frac{1}{2}hc)^2-\frac{1}{4}h^2c^2}{h^2+c^2}}.$$

Plug into the expression for V_{c+p} : $h = a\cdot c$

$$V_{c+p} = c^3 + \frac{1}{6}a\cdot c\sqrt{\frac{(S-6c^2+\frac{1}{2}a\cdot c^2)^2-\frac{1}{4}a^2c^4}{a^2+1}}.$$

As usual, make the expression for V_{c+p} with S, after dropping the primes

$$V_{c+p} = c^3 + \frac{1}{6}a\cdot c\sqrt{\frac{[1-\left(6-\frac{1}{2}a\right)c^2]^2-\frac{1}{4}a^2c^4}{a^2+1}}.$$

The square root $\sqrt{\frac{[1-\left(6-\frac{1}{2}a\right)c^2]^2-\frac{1}{4}a^2c^4}{a^2+1}}$ leads to the constraint:

$$[1-\left(6-\frac{1}{2}a\right)c^2]^2-\frac{1}{4}a^2c^4 > 0,$$

or

$$1-(12-a)c^2+\left[\left(6-\frac{1}{2}a\right)c^2\right]^2-\frac{1}{4}a^2c^4 > 0 \rightarrow 1-(12-a)c^2+(36-6a)c^4+\frac{1}{4}a^2c^4-\frac{1}{4}a^2c^4 > 0 \rightarrow c^4-\frac{12-a}{36-6a}c^2+\frac{1}{36-6a} > 0.$$

Then,

$$c^2 > \frac{6-\frac{1}{2}a}{36-6a} \pm \sqrt{(\frac{6-\frac{1}{2}a}{36-6a})^2-\frac{1}{36-6a}} \rightarrow c^2 > \frac{6-\frac{1}{2}a}{36-6a} \pm \frac{\frac{1}{2}a}{36-6a}.$$

Since $0 < a < 1$, the constraint is

$$c > \sqrt{\frac{1}{6-a}}.$$

I use the graphical approach to find out about the heat exchange. By making the variables dimensionless using S, makes the graphical approach feasible since $h = a\cdot c = O(1)$, and $c = O(1)$. A few graphs are plotted.

A graph of V_{c+p} with $a = \frac{1}{4}$:

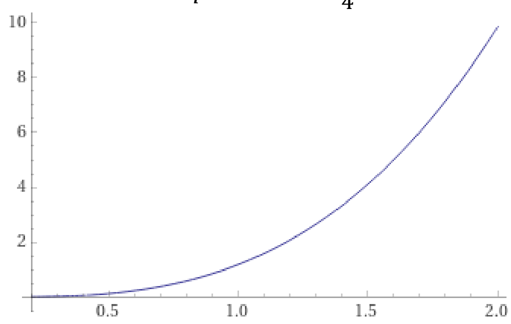


Figure 4.15 V_{c+p} as a function of c with a=1/4

With increasing c , heat exchange decreases since the volume increases for a given surface S. Consequently, to improve heat exchange c decreases.

Next, a graph of V_{c+p} with $a = \frac{1}{2}$:

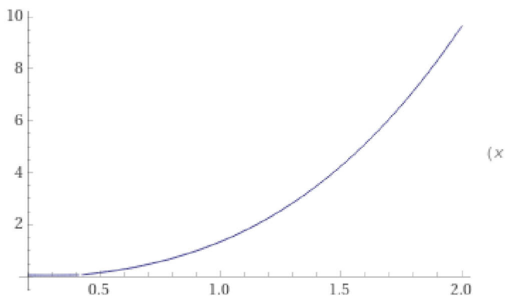


Figure 4.16 V_{c+p} as a function of c with a=1/2

The same pattern as in Figure 4.15.

A graph of V_{c+p} with $a = 1$:

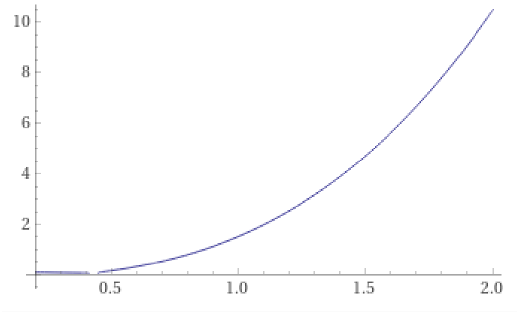


Figure 4.17 V_{c+p} as a function of c with a=1

The same pattern as in Figure 4.15.

The constraint $c > \sqrt{\frac{1}{6-a}}$ is illustrated in the above Figure.

When $\sqrt{\frac{1}{6}} < c < \sqrt{\frac{1}{6-a}}$, V_{c+p} is a complex number. Hence a meaningful value of c for improving heat exchange with the environment is $c > \sqrt{\frac{1}{6-a}}$.Then, for these values of c heat exchange increases with decreasing c.

§ 4.5 Optimize the Sphere and the Pyramid

§ 4.5.1 Optimize the Surface S for a given Volume V

For the analysis of the sphere and the cylinder I use some of the results of the sphere and the cone, see Figure 3.1 page 14:

$$A = 2\pi\left(r_{sp}^2 - r_{sp}\sqrt{r_{sp}^2 - r_c^2}\right),$$

where r_c is the radius of the cylinder.

$$V_{sh} = \frac{\pi}{3}\left[2r_{sp}^3 - \left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right].$$

Then, the surface of primal body and the cylinder

$$S = 4\pi r_{sp}^2 + 2\pi r_c L - 2\pi\left(r_{sp}^2 - r_{sp}\sqrt{r_{sp}^2 - r_c^2}\right) + \pi r_c^2 =$$

$$= 2\pi r_{sp}^2 + 2\pi r_c L + 2\pi r_{sp}\sqrt{r_{sp}^2 - r_c^2} + \pi r_c^2.$$

The volume

$$V = \frac{4}{3}\pi r_{sp}^3 + \pi r_c^2 L - \frac{\pi}{3}\left[2r_{sp}^3 - \left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right] =$$

$= \frac{2}{3}\pi r_{sp}^3 + \pi r_c^2 L + \frac{\pi}{3}\left[\left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right].$

where L is the length of the cylinder.

The surface S is dependent on three independent variables: r_{sp}, r_c, and L.

Since we analyse the case for a given volume, L can be expressed in V:

$$L = \frac{V - \frac{2}{3}r_{sp}^3 - \frac{1}{3}\left[\left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right]}{r_c^2}.$$

The above expression for L is plugged into the expression for S:

$$S = 2\pi r_{sp}^2 + 2\pi r_c L + 2\pi r_{sp}\sqrt{r_{sp}^2 - r_c^2} + \pi r_c^2 \Longrightarrow$$

$$\Longrightarrow S = 2\pi r_{sp}^2 + 2\pi \frac{V - \frac{2}{3}r_{sp}^3 - \frac{1}{3}\left[\left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right]}{r_c} + 2\pi r_{sp}\sqrt{r_{sp}^2 - r_c^2} + \pi r_c^2.$$

Again, we make the preceding expression dimensionless with V:

$S' = \frac{S}{V^{2/3}}$, $r'_{sp} = \frac{r_{sp}}{V^{1/3}}$, and $r'_c = \frac{r_c}{V^{1/3}}$.

Then after dropping the primes the expression for the surface reads

$$S = 2\pi r_{sp}^2 + 2\pi \frac{1 - \frac{2}{3}r_{sp}^3 - \frac{1}{3}\left[\left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right]}{r_c} + 2\pi r_{sp}\sqrt{r_{sp}^2 - r_c^2} + \pi r_c^2 =$$

$$= 2\pi r_{sp}^2 + 2\pi \frac{1 - \frac{2}{3}r_{sp}^3 - \frac{1}{3}\left[\left(2r_{sp}^2 - 3r_{sp}r_c + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right]}{r_c} + \pi r_c^2.$$

For a given V , L becomes negative by increasing r_{sp} for a given value of r_c.

The dimensionless function of L

$$L = \frac{1 - \frac{2}{3}r_{sp}^3 - \frac{1}{3}\left[\left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right]}{r_c^2}.$$

In the plot below this is illustrated

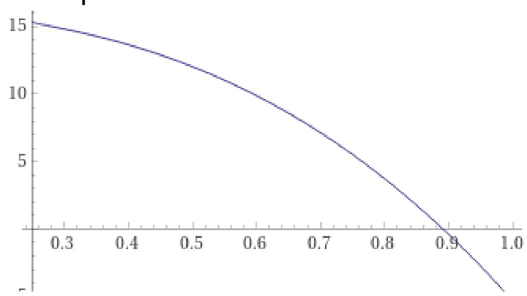


Figure 4.18 L as a function of the radius of the sphere with r_c =0.25

L becomes negative for $r_{sp} > 0.89$.

Next, a plot of S with $r_c = 0.25$:

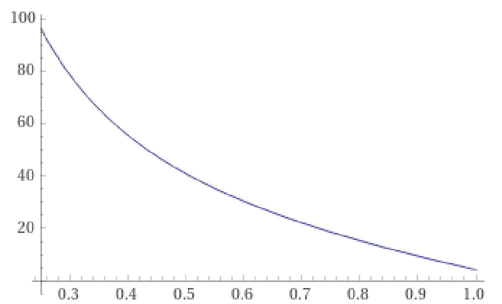


Figure 4.19 S as a function of the radius of the sphere with r_c =0.25

Improving heat exchange with the environment is obtained by reducing the radius of the sphere r_{sp}. The length of the cylinder increases simultaneously. See Figure 4.18.
As a special case I analyse the $r_c = r_{sp}$. It looks a bit like the toucan.

Figure 4.20 The Toucan

With $r_c = r_{sp}$:

$S = 3\pi r_c^2 + 2\pi \frac{1-\frac{2}{3}r_c^3}{r_c}$,

and

$L = \frac{1-\frac{2}{3}r_c^3}{r_c^2}$

In the figure below S is presented as a function of the radius of the beak.

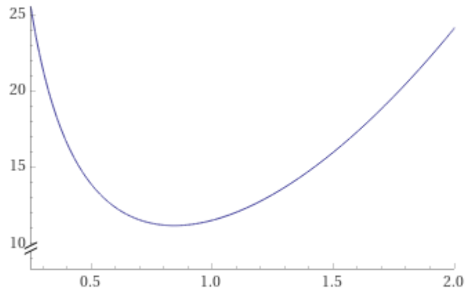


Figure 4.21 S as a function of the radius of the beak

Left of the minimum improvement of heat exchange is obtained in a more efficient way than at the right of the minimum($r_c \cong 0.84$).
The length of the beak becomes 0 at $r_c \cong 1.14$.

§ 4.5.2 Optimize the Volume V for a given Surface S

We use the Toucan case: the radius of the sphere equal to the radius of the cylinder(beak).
The surface, with $r_{sp} = r_c$:

$S = 2\pi r_{sp}^2 + 2\pi r_c L + 2\pi r_{sp}\sqrt{r_{sp}^2 - r_c^2} + \pi r_c^2 = 3\pi r_c^2 + 2\pi r_c L.$

The volume

$V = \frac{2}{3}\pi r_{sp}^3 + \pi r_c^2 L + \frac{\pi}{3}\left[\left(2r_{sp}^2 + r_c^2\right)\sqrt{r_{sp}^2 - r_c^2}\right] = \frac{2}{3}\pi r_c^3 + \pi r_c^2 L.$

With a given surface

$L = \frac{S-3\pi r_c^2}{2\pi r_c}$.

Then we have for the volume

$V = \frac{2}{3}\pi r_c^3 + r_c \frac{S - 3\pi r_c^2}{2} = \frac{r_c}{2} S - \frac{5}{6}\pi r_c^3$.

Make the expressions for

L and V dimensionless with S

$L' = \frac{L}{S^{1/2}}, V' = \frac{V}{S^{3/2}} \rightarrow r_c' = \frac{r_c}{S^{1/2}}$.

After dropping the primes:

$V = \frac{r_c}{2} - \frac{5}{6}\pi r_c^3$,

and

$L = \frac{1 - 3\pi r_c^2}{2\pi r_c}$.

A plot of V:

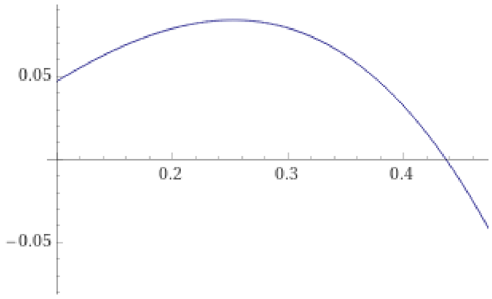


Figure 4.22 V as a function of the radius of the beak

By decreasing the volume left of the maximum($r_c \cong 0.25$) , the ratio surface to volume is increased and the heat exchange with the environment is improved.

A plot of L:

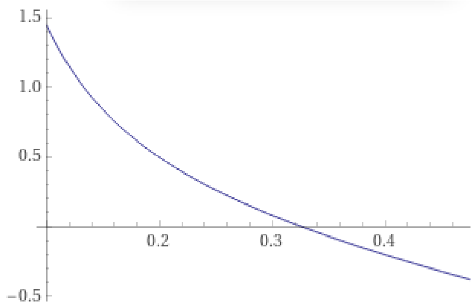


Figure 4.23 The length of the beak as a function of its radius

As expected, the length of the beak increases with decreasing radius of the beak.

$L = 0$ for $r_c \cong 0.33$.

§ 5 Conclusions

As mentioned in the introduction, adaptation to climate change, according to the research of Ryding, et all, is about altering the bodies of animal species by increasing the dimensions of appendages. Furthermore, Ryding, et all, page 4: "*While studies on morphological effects of climate change in mammals relate to overall body size, changes in appendage size are evident, too.*"

In this paper the effect of increase of the surface of the primal body on heat exchange with the environment is analysed in § 2. The effect is clear.
The modelling in this paper describes the adaptation of animal bodies to climate change. By increasing the surface of the appendages, as analysed in § 3, the surface of the animal increases relative to the total volume. So does the heat exchange with the environment. Use has been made of analytical geometry.
In § 4, we analysed the changes of the geometry of the primal body and the appendages. The heat exchange with the environment is improved by changing the morphology. Obviously, up to a point. The point is the relation between the temperature of the body and the wet bulb temperature of the environment. *Wet bulb conditions occur when heat and humidity are too high for sweat to evaporate*, www.insider.com .
The drama that follows is described by Brannen(2017).
In the table 1 below, I summarize the effect of increasing the surface of the appendages. The primal body is treated to be constant.

Table 1: Summary of the Results

Primal Body	Appendages	Increasing Heat Exchange
Sphere § 3.1	Limbs	+
Cylinder § 3.2	Limbs	+
Sphere §§ 3.3,3.4 and 3.5	Beak	+

In table 2 the morphological adaptation of the combination of primal body and appendages are summarized.

Table 2 : Summary of the Results.

§ 4.1 Sphere and four limbs	Adaptation is possible
§ 4.2 Cylinder and four limbs	idem
§ 4.3 Sphere and beak(cone)	idem
§ 4.4 Sphere and beak(pyramid)	idem
§ 4.5 Sphere and beak(cylinder)	idem

§ 5 Literature

Brannen, P, *The Ends of the World: Volcanic Apocalypses, Lethal Oceans, and Our Quest to Understand Earth's Past Mass Extinctions*, HarperCollins, 2017.

Luttwak, E. N., *The Grand Strategy Of The Roman Empire. From the First Century A.D. to the Third,"* The Johns Hopkins University Press, Baltimore, and London, 1979.

Noordzij, L.(1) , *Isoperimetric Inequality and Isoepifaeic Inequality.* www.leennoordzij.me 2021.

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Ryding, S., et all, *Shapeshifting: changing animal morphologies as a response to climate warming, Trends I Ecology & Evolution*, September 7th 2021.

The Economist, *Climate change and evolution, A warmer planet is changing how animals look*, section on Science and Technology, September 11th 2021.

www.en.wikipedia.org , Solid Angle, September 15th 2021.

Appendix A function of Two Independent Variables

A1 The Sphere as a Primal Body and the Limbs Section 3.1

The ratio of the body surface and the volume with four limbs

$ratio_B = \frac{4\pi r_{sh}^2+8\pi r_{cl}L}{\frac{4}{3}\pi r_{sh}^3+4\pi r_{cl}^2 L}$.

Hence,

$ratio_B = ratio_B(r_{cl}, L)$.

First, I make $ratio_B$ dimensionless with r_{sh}, the given radius of the sphere, i.e., the primal body:

$x = \frac{r_{cl}}{r_{sh}}, y = \frac{L}{r_{sh}}$.

Then the new function $ratio_B$ is $F(x,y) = r_{sh} \cdot ratio_B$:

$F(x,y) = \frac{1+2xy}{\frac{1}{3}+x^2 y}$.

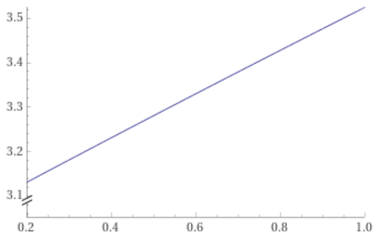

Figure 1A F as a function of y with x=0.1

In Figure 1A, a plot of $F(x,y)$ for $x = 0.1$. The ratio increases with y , illustrating the increase of heat exchange. Over the range of y : an increase of about 15%.

The derivatives:

$-\frac{\partial F}{\partial x} = \frac{2y}{\frac{1}{3}+x^2y} - \frac{1+2xy}{\left(\frac{1}{3}+x^2y\right)^2}2xy = 0 \Longrightarrow 2y\left(\frac{1}{3}+x^2y\right) - (1+2xy)2xy = 0 \Longrightarrow$

$\Longrightarrow \frac{2}{3}y + 2x^2y^2 - 2xy - 4x^2y^2 = 0 \Longrightarrow \frac{1}{3}y - xy - x^2y^2 = 0 = y\left(\frac{1}{3} - x - x^2\right),$

with $y = 0$, and

$\frac{1}{3} - x - x^2 y = 0$.

$-\frac{\partial F}{\partial y} = \frac{2x}{\frac{1}{3}+x^2y} - \frac{1+2xy}{\left(\frac{1}{3}+x^2y\right)^2}x^2 = 0 \Longrightarrow 2x\left(\frac{1}{3}+x^2y\right) - (1+2xy)x^2 = 0 \Longrightarrow$

$\Longrightarrow \frac{2}{3}x + 2x^3y - x^2 - 2x^3y = 0 = x\left(\frac{2}{3} - x\right),$

with $x = 0$, and

$\frac{2}{3} - x = 0$.

Then,

$y\left(\frac{1}{3} - x - x^2\right) = 0,$

with $y = 0$, and $x = 0$
$\frac{1}{3} - \mathrm{x} - x^2\mathrm{y} = 0 \Longrightarrow y = -\frac{3}{4}$, and $y = \frac{1}{3}$.
So, we obtain four stationary points: $(0{,}0), \left(0, \frac{1}{3}\right), (\frac{2}{3}, 0)$ and $(\frac{2}{3}, -\frac{3}{4})$.
The latter stationary point has no physical meaning.
Furthermore,
F at $(0{,}0), \left(0, \frac{1}{3}\right), (\frac{2}{3}, 0)$: $F = 3 \Longrightarrow$ no appendages: the sphere.

In section § 3.1, we found:
$\frac{2}{3} > \frac{r_{cl}}{r_{sh}}$.
So the maximum value for $\frac{r_{cl}}{r_{sh}} = \frac{2}{3}$.
Now, I will illustrate the importance of $\frac{r_{cl}}{r_{sh}} = \frac{2}{3}$.
For this $F(x, y) = r_{sh} \cdot ratio_B = \frac{1+2xy}{\frac{1}{3}+x^2y}$ is plotted below for two cases: $x = 0.65$, and $x = 0.7$.

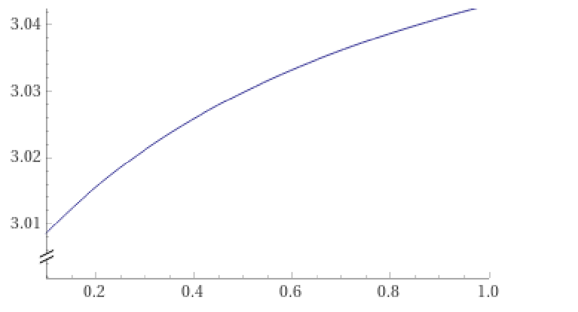

Figure 2A F as a function of y with x=0.65

In the figure above we have $3.01 < F < 3.04$.

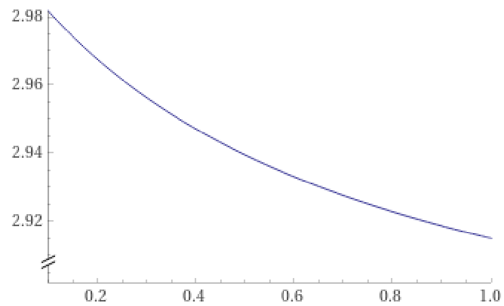

Figure 3A F as a function of y with x=0.7

In the figure above we have 2.92$< F < 2.98$.
The two plots above 2A and 3A illustrates the change of an increasing F(ratio) into a decreasing F(ratio)$\Longrightarrow$ a change from increasing heat exchange into a decreasing heat

exchange. So, $x = \frac{r_{cl}}{r_{sh}} = \frac{2}{3}$ is a threshold.[7]
Here, I assumed this to be a maximum.
A proof.
We need:

- $\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2y}{\frac{1}{3}+x^2y}\right) - \frac{\partial}{\partial x}\left[\frac{1+2xy}{\left(\frac{1}{3}+x^2y\right)^2}2xy\right] =$

$$= -\frac{2y}{\left(\frac{1}{3}+x^2y\right)^2}2xy - \frac{2y+8xy^2}{\left(\frac{1}{3}+x^2y\right)^2} + 2\frac{1+2xy}{\left(\frac{1}{3}+x^2y\right)^3}4x^2y^2 =$$

$$= -\frac{2y+12xy^2}{\left(\frac{1}{3}+x^2y\right)^2} + 8\frac{x^2y^2+2x^3y^3}{\left(\frac{1}{3}+x^2y\right)^3}.$$

- $\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial x}\left(\frac{2x}{\frac{1}{3}+x^2y}\right) - \frac{\partial}{\partial x}\left[\frac{1+2xy}{\left(\frac{1}{3}+x^2y\right)^2}x^2\right] =$

$$= \frac{2}{\frac{1}{3}+x^2y} - \frac{4x^2y}{\left(\frac{1}{3}+x^2y\right)^2} - \frac{2x+4x^2y}{\left(\frac{1}{3}+x^2y\right)^2} + 2\frac{1+2xy}{\left(\frac{1}{3}+x^2y\right)^3}2x^3y =$$

$$= \frac{2}{\frac{1}{3}+x^2y} - \frac{2x+8x^2y}{\left(\frac{1}{3}+x^2y\right)^2} + 4\frac{x^3y+2x^4y^2}{\left(\frac{1}{3}+x^2y\right)^3}.$$

- $\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2x}{\frac{1}{3}+x^2y}\right) - \frac{\partial}{\partial y}\left[\frac{1+2xy}{\left(\frac{1}{3}+x^2y\right)^2}x^2\right] = -\frac{2x^3}{\left(\frac{1}{3}+x^2y\right)^2} - \frac{2x^3}{\left(\frac{1}{3}+x^2y\right)^2} + 2\frac{1+2xy}{\left(\frac{1}{3}+x^2y\right)^3}x^4 =$

$$= -\frac{4x^3}{\left(\frac{1}{3}+x^2y\right)^2} + 2\frac{x^4+2x^5y}{\left(\frac{1}{3}+x^2y\right)^3}.$$

With these ingredients we can construct the Hessian matrix:

$$M = \begin{pmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{pmatrix}.$$

The determinant of the matrix:

$$\begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial y} \\ \frac{\partial^2 F}{\partial y \partial x} & \frac{\partial^2 F}{\partial y^2} \end{vmatrix} = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 F}{\partial y \partial x},$$

and the trace of the matrix:

$$TrM = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2}.$$

$x = 0$, and $y = 0$.

Plug these values into the determinant and the trace. The result is:

- the determinant is negative: $\frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial x \partial y}\frac{\partial^2 F}{\partial y \partial x} = -12$,
- the trace is zero: $\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} = 0$.

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[7] The plots do not change much when we choose $x = 0.666$ and $x = 0.667$. However, $3.001 < F < 3.002$ and $2.999 < F < 3.000$, respectively. Near the stationary point $F = 3$.

The determinant is negative, so irrespective of the trace , the stationary point is a saddle point at $(x, y) = (0,0)$.

Now, I plot $F(x, y)$ with $y = 1$:

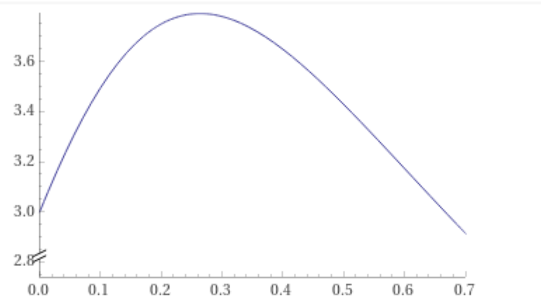

Figure 4A F(x,y) as a function of x with y=1

In Figure 4A $F_{max} = 3.75$ at $x = 0.28$.

With $y = 0.5$

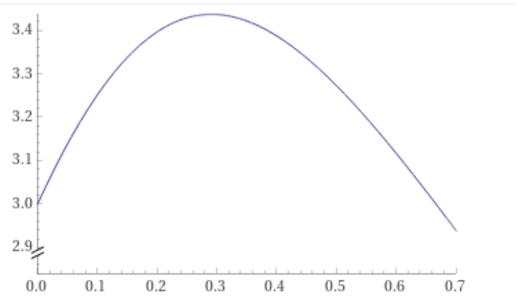

Figure 5A F(x,y) as a function of x with y=0.5

In Figure 5A $F_{max} = 3.42$ at $x = 0.29$.

With $y = 1.5$:

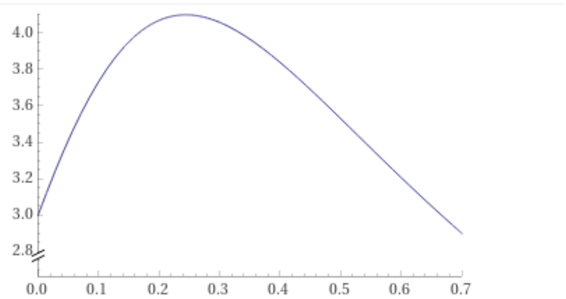

Figure 6A F(x,y) as a function of x with y=1.5

In Figure 6A $F_{max} = 4.1$ at $x = 0.24$.

With $y = 2$:

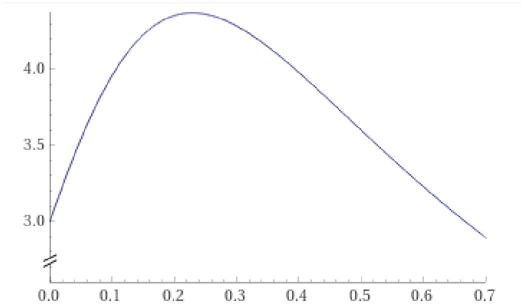

Figure 7A F(x,y) as a function of x with y=2

In Figure 7A $F_{max} = 4.8$ at $x = 0.22$.

In Figure 8A, I pictured $F(x, y)$ with $0 < x < 0.7$ and $0 < y < 1.5$.
Th five maxima of F are indicated.

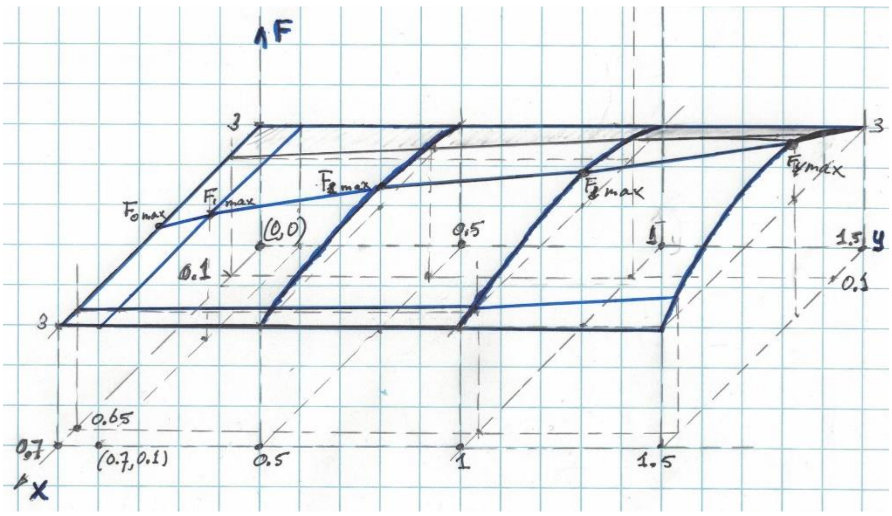

Figure 8A F(x,y) pictured as a function of 0<x<0.7, and 0<y<1.5

I calculated F_{max} :
$y = 3$: $F_{max} = 4.9$ at $x = 0.22$,
$y = 4$: $F_{max} = 5.3$ at $x = 0.2$.
With increasing y, F_{max} increases and the x- coordinate decreases.
Let us plot $F(x, y)$ as a function of x with $y = 20$, a not very realistic geometry.

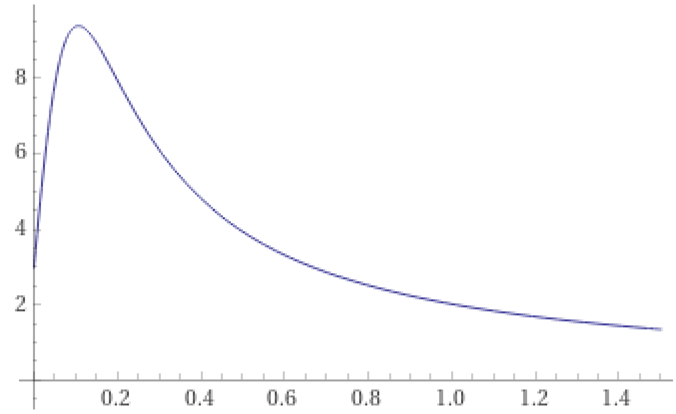

Figure 9A F(x,y) as a function of x with y=20

In Figure 9A:
$y = 20$: $F_{max} = 9.5$ at $x = 0.12$.
$F(x, y) = \frac{1+2xy}{\frac{1}{3}+x^2y} = \frac{\frac{1}{y}+2x}{\frac{1}{3y}+x^2}$.
Hence, for $y \gg 1$: $F(x, y) \cong \frac{1}{x}$. What does a 3-D picture of $F(x, y)$ look like?

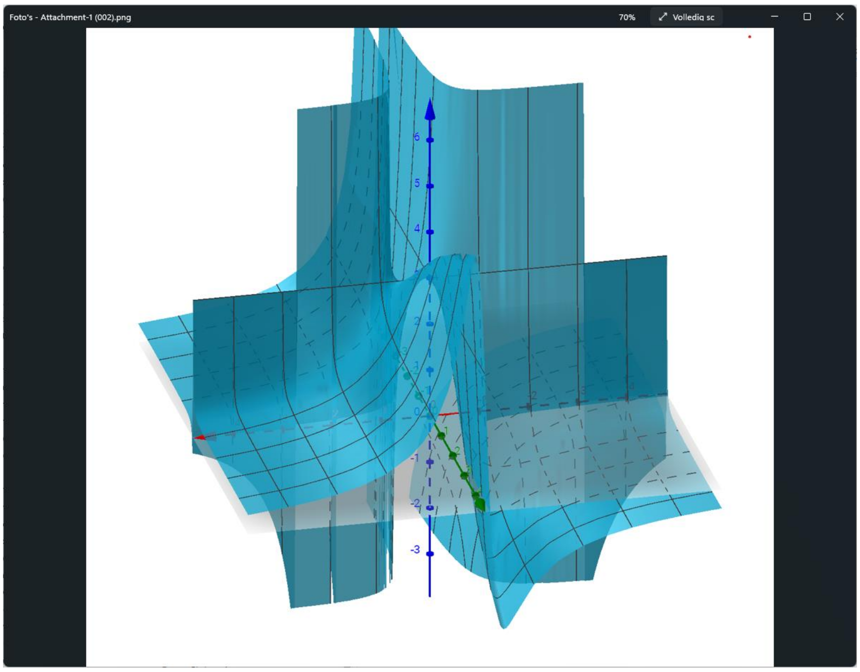

Figure 10 A F(x,y) in 3-D with GeoGebra

In Figure 10A I showed $F(x, y)$ in 3-D. In the analysis the relevant part is the left corner below with $x > 0$, the red arrow, and $y > 0$, the green arrow. F is represented by the vertical, blue, arrow.

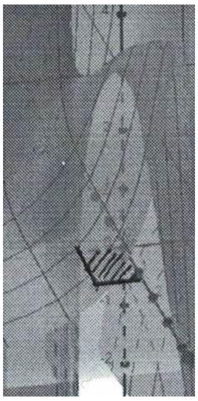

Figure 11A A detail of Figure 10A

To put the analysis in perspective, I showed in Figure 11A the range for $0 < x < 1$, and $0 < y < 1.5$.

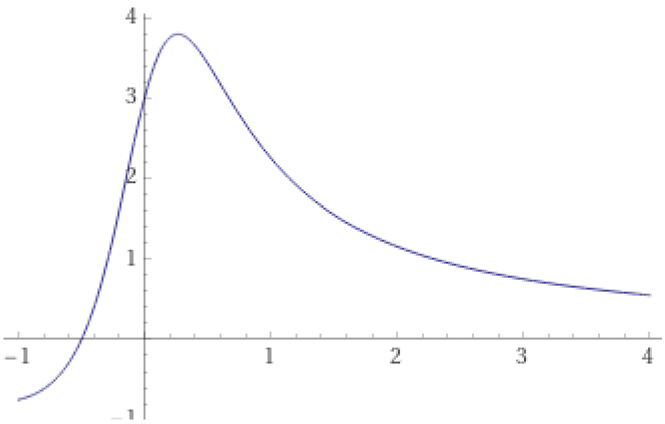

Figure 12A F(x,y) as a function of x with y=1.

Furthermore, in Figure 12A a cross section of F is shown for $y = 1$.