Flooding, Climate Change, and a Model.

Update 2022-12-29. Introduction: remark on numerical computation.

Dr.l.noordzij@leennoordzij.nl
www.leennoordzij.me

Content

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Introduction.

In The Economist Sept 2nd 2017 a Briefing is presented on Flooding and how to deal with it. In the section The Chances of Disaster and Frequency Modulation of this Briefing the author wrote: " *The scars left by the biggest past events provide benchmarks for what might happen again ………*."

In the past the Netherlands has been prone to heavy flooding. Salt water replaced fresh water and vice versa. Such a sequence can be modelled. With present days measurement techniques, the impact of this flooding can be estimated.

Let us give it a try. The idea is based on work I did as an undergraduate at The University of Twente in the Netherlands. The title of this consignment is: *A Diffusion Problem*- memorandum of research instruction, February 1968.

It is about exact solutions of the diffusion equation. Exact solutions exist when using some approximations. Now numerical methods are available. However, the method presented in this paper can be useful to start the process of numerical computation.

Description of the diffusion model

At a certain known point of time, $t = 0$ the flooding replaces fresh water by saltwater with an unknown salt concentration X . During a period t_1 started at $t = 0$, X is assumed to be constant. The salt is absorbed – diffused – by the bottom/soil. At the end of this time period the salt concentration of the water became zero in a brief period of time. In the model this period is set equal to zero. Then, desalination of the bottom started from t_1 until a second time t_2. At the latter time a new flooding makes the water salty again- in a brief time period set equal to zero- with a known constant salt concentration A. The salty water with concentration A is still present.

We assume the salt concentration in the bottom to be described by the diffusion equation. The model is chosen to be one-dimensional: so $\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \ll \frac{\partial}{\partial z}$.

The question to be answered is: is it possible to estimate X by measuring the salt concentration in the bottom/soil? Measurements made at the present time.

Solution

The first period $0 < t < t_1$:

the diffusion equation $\frac{\partial C_1}{\partial t} = D \frac{\partial^2 C_1}{\partial z^2}$, a linear differential equation where C_1 denotes the salt concentration in the bottom $z > 0$ and D the diffusion coefficient. This coefficient is considered to be a constant.

The initial and boundary conditions are:

$C_1(z,0) = 0$ for $z > 0$, and

$C_1(0,t) = X$, X is the unknown salt concentration we want to find out about.

In addition the concentration for $z \to \infty$ is equal to zero.

The solution for this time period is: $C_1(z,t) = X\mathrm{erfc}(\frac{z}{2\sqrt{Dt}})$, (1)

the erfc is defined by $\mathrm{erfc}s = \frac{2}{\sqrt{\pi}}\int_s^\infty e^{-r^2}\,dr$.

This solution of the diffusion equation can be found in standard textbooks on heat- and mass transfer. Laplace transformation is part of the toolkit.

The second period $t_1 \le t < t_2$:

the salt water is pushed aside instantaneously by fresh water.

The diffusion equation $\frac{\partial C_2}{\partial t} = D \frac{\partial^2 C_2}{\partial z^2}$, a linear differential equation where C_2 denotes the salt concentration in the bottom $z > 0$.

The initial condition is:

$C_2(z,0) = C_1(z,t_1)$ for $z > 0$, and

the boundary condition is given by the mass transfer into the freshwater driven by the concentration difference between the bottom and the water at $z = 0$:

$D\frac{\partial C_2}{\partial z} = kC_2$, where k is the mass transfer coefficient and the subsequent concentration of salt in the fresh water is neglected. The coefficient k is also considered to be a constant.

The solution for this time period is:

$$C_2(z,t) = \int_0^\infty C_1(\beta,t_1)[\frac{1}{2\sqrt{\pi Dt}}\{e^{\frac{-(\beta+z)^2}{4Dt}} + e^{\frac{-(\beta-z)^2}{4Dt}}\} - \frac{k}{D}\{e^{\frac{k(\beta+z)+k^2t}{D}}\mathrm{erfc}(\frac{\beta+z}{2\sqrt{Dt}} + \frac{k\sqrt{t}}{\sqrt{D}})\}]d\beta, \quad (2)$$

and with Eq.(1):

$C_1(\beta,t_1) = X\mathrm{erfc}(\frac{\beta}{2\sqrt{Dt_1}})$.

The third period $t_2 \le t$:

the fresh water is instantaneously replaced by salty water with a known salt concentration A.

The diffusion equation reads $\frac{\partial C_3}{\partial t} = D\frac{\partial^2 C_3}{\partial z^2}$, where C_3 denotes the salt concentration in the bottom $z > 0$.

The initial and boundary conditions are:
$C_3(z,0) = C_2(z,t_2)$ for $z > 0$, and
$C_3(0,t) = A$.
The solution for this time period is:

$$C_3(z,t) = A\text{erfc}\left(\frac{z}{2\sqrt{Dt}}\right) + \int_0^{\infty} C_2(m,t_2)[\frac{1}{2\sqrt{\pi Dt}}\{e^{\frac{-(m-z)^2}{4Dt}} - e^{\frac{-(m+z)^2}{4Dt}}\}]dm, \quad (3)$$

with
$C_2(m,t_2)$, given by Eq.(2), m substituted for β and t_2 for t.
Eq. (3) is the one we were looking for.

Discussion

The result we were looking for is represented by Eq. (3).
For given values of D, k, A, t_1, t_2 and t, the expression represented by Eq. (3) basically reads:

$$C_3(z,t) = A\text{erfc}(\alpha z) + Xf(z), \quad (4)$$

where α is a constant and $Xf(z)$ represents the second term on the right hand side in Eq. (3).
$C_3(z,t)$ is found from measurements and the unknown X can be calculated. Theoretically, X is a constant for any value of $z > 0$.
Executing various measurements in a horizontal grid you get an idea how far the salty water penetrated at time $t = 0$.
Final remarks: we made a lot of assumptions like the diffusion coefficient to be a constant. Well, for diffusion coefficient dependent on the salt concentration in the bottom the diffusion equation changes into a second order non-linear differential equation. So, for a constant diffusion coefficient:

$$\frac{\partial D}{\partial z} \ll \frac{\partial C}{\partial z}.$$

Conclusion

"The scarce left by the biggest past events……….. provide marks what might happen again."