Isoperimetric Inequality and the Isoepifaeic Inequality
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Tags: isoperimetric inequality, isoepifaeic inequality

§ 1 Introduction

In "Puzzles with my grandchildren" (www.leennoordzij.me) I addressed the problem of finding the maximum area A for a given perimeter of length L in a cartesian frame. The circle creates the largest area.

There is a very practical application of the isoperimetric inequality. In "The Grand Strategy Of The Roman Empire" (E. N. Luttwak): *'….A second clear-cut difference is the ground plans of late Roman fortifications. Old-style rectangles with rounded ditch defences naturally persisted, since in many cases old fortifications remained in use, but the square layout became predominant, together with irregular quadrilaterals (Yverdon), rough circles (Jünkerath) and bell-shapes-where the broader side rested on a river or the sea (Koblenz, Altenburg, Solothurn, Altrip). The advantage of proximate circles and proximate squares over the older rectangular pattern is, as noted before, the shorter length of the wall for any given internal area. The perfect circle-theoretically optimal-was normally avoided because it was difficult to build…..'*

On the internet quite a few references can be found for the isoperimetric inequality. Without any preferences I mention a paper by Hehl and another by Treibergs. From these papers you can learn that grandchildren need some basic understanding of mathematics. These papers are attractive since you can find a lot of historical evidence on the isoperimetric inequality. Archimedes is mentioned, but alas no historical material is available.

The isoperimetric inequality is a 2-dimensional inequality. Is there a 3-D equivalent? In Noordzij, *Adaptation to Climate via Morphological Change and Optimizing Heat Exchange*, attention is paid to the subject matter. The 3-D equivalent is to find the maximum mass/volume for a given surface of that mass/volume. This inequality can be used to study the effect of climate change, i.e., raising temperatures, on the morphology of vertebrates: adapting to climate change. To improve heat exchange with the environment, this adaptation resulted into an increasing surface of the vertebrate for given/mass.

The equivalent with the isoperimetric inequality is denoted isoepifaeic inequality.

In § 2 some examples of the isoperimetric inequality are presented. A proof of the isoperimetric inequality can be found in § 3. The isoepifaeic inequality is illustrated with a couple of three-dimensional structures in § 4. The proof of the isoepifaeic inequality is presented in § 5.

§ 2 The Isoperimetric Inequality; Some Examples

The inequality reads: $A \leq \frac{L^2}{4\pi}$ (Hehl, Treibergs),

where A is the area with perimeter L . See also www.en.wikipedia.org .

Then, for a circle with radius $R = L$ the equality holds: $A = \pi R^2 = \frac{(2\pi R)^2}{4\pi}$.

§ 2.1 Triangles with Perimeter L

We start with finding the maximum area of a triangle with perimeter L.

For the area A_t of the triangle I will use Heron's formula:

$$A_t = \sqrt{\frac{L}{2}(\frac{L}{2} - a)(\frac{L}{2} - b)(\frac{L}{2} - c)}\ ,$$

where $L = a + b + c$,
and a, b, c are the sides of the triangle.
I substitute in the above formula of A_t: $a = L - b - c$.
Consequently

$$A_t = \sqrt{\frac{L}{2}(b + c - \frac{L}{2})(\frac{L}{2} - b)(\frac{L}{2} - c)}.$$

Hence, A_t is a function of (b, c) with parameter L.
What do we learn from
$\frac{\partial A_t}{\partial b} = 0$, and $\frac{\partial A_t}{\partial c} = 0$ in order to obtain the maximum value?

$$\frac{\partial A_t}{\partial b} = \frac{1}{2}\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}(b+c-\frac{L}{2})(\frac{L}{2}-b)(\frac{L}{2}-c)}}[\left(\frac{L}{2} - b\right)\left(\frac{L}{2} - c\right) - \left(b + c - \frac{L}{2}\right)\left(\frac{L}{2} - c\right)],$$

and

$$\frac{\partial A_t}{\partial c} = \frac{1}{2}\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}(b+c-\frac{L}{2})(\frac{L}{2}-b)(\frac{L}{2}-c)}}[\left(\frac{L}{2} - b\right)\left(\frac{L}{2} - c\right) - \left(b + c - \frac{L}{2}\right)\left(\frac{L}{2} - b\right)].$$

First , with $\frac{\partial A_t}{\partial b} = 0$, we find an expression for b:

$$b = \frac{\frac{3Lc}{2} - c^2 - \frac{L^2}{2}}{2c - L} = -\frac{(c-L)\left(c-\frac{L}{2}\right)}{2\left(c-\frac{L}{2}\right)} = -\frac{1}{2}(c - L) \,,$$

with $L > 2c$[1].
Next, with $\frac{\partial A_t}{\partial c} = 0$, we find an expression for c:

$$c = \frac{\frac{3Lb}{2} - b^2 - \frac{L^2}{2}}{2b - L} = -\frac{(b-L)\left(b-\frac{L}{2}\right)}{2\left(b-\frac{L}{2}\right)} = -\frac{1}{2}(b - L) \,,$$

with $L > 2b$.
After some algebra, these two expressions for b and c results into:
$b = c$.
So, the maximum we found for this equality is an isosceles triangle.
Did we find a maximum? Let us find out.
For a function of two variables we can do the analysis.
Then we need the second derivatives:
$\frac{\partial^2 A}{\partial b^2}, \frac{\partial^2 A}{\partial c^2}, \frac{\partial^2 A}{\partial b \partial c}$, and $\frac{\partial^2 A}{\partial c \partial b}$.
Here I left out the subscript t, triangle, of A_t for convenience.
We make use of the Hessian matrix:

$$H = \begin{pmatrix} \frac{\partial^2 A}{\partial b^2} & \frac{\partial^2 A}{\partial b \partial c} \\ \frac{d^2 A}{\partial c d b} & \frac{\partial^2 A}{\partial c^2} \end{pmatrix}.$$

Next, the determinant of the matrix:

[1] For $L \leq 2c$ there is no triangle left.

$$Det\ H = \begin{vmatrix} \frac{\partial^2 A}{\partial b^2} & \frac{\partial^2 A}{\partial b \partial c} \\ \frac{\partial^2 A}{\partial c \partial b} & \frac{\partial^2 A}{\partial c^2} \end{vmatrix} = \frac{\partial^2 A}{\partial b^2}\frac{\partial^2 A}{\partial c^2} - \frac{d^2 A}{\partial b \partial c}\ \frac{\partial^2 A}{\partial c \partial b}.$$

The trace of the matrix:

$$TrH = \frac{\partial^2 A}{\partial b^2} + \frac{\partial^2 A}{\partial c^2}.$$

If the determinant is positive and the trace of the matrix is negative we have for

$\frac{\partial A}{\partial b} = 0$, and $\frac{\partial A}{\partial c} = 0$,

a local maximum.

First,

$$-\frac{\partial^2 A}{\partial b^2} = -\frac{1}{2}\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)}}[L-2c] - \frac{1}{4}\frac{(\frac{L}{2})^2[\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-c\right)]^2}{[\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)]^{3/2}},$$

$$-\frac{\partial^2 A}{\partial c^2} = -\frac{1}{2}\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)}}(L-2b) - \frac{1}{4}\frac{(\frac{L}{2})^2[\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)]^2}{[\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)]^{3/2}}.$$

With $\frac{\partial A}{\partial b} = 0$, and $\frac{\partial A}{\partial c} = 0, \Longrightarrow b = c \Longrightarrow$

$$-\frac{\partial^2 A}{\partial b^2} = -\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}\left(2b-\frac{L}{2}\right)}} - \frac{1}{4}\frac{(\frac{L}{2})^2[L-3b]^2}{\left(\frac{L}{2}-b\right)[\frac{L}{2}\left(2b-\frac{L}{2}\right)]^{3/2}},$$

$$-\frac{\partial^2 A}{\partial c^2} = -\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}\left(2b-\frac{L}{2}\right)}} - \frac{1}{4}\frac{(\frac{L}{2})^2[L-3b]^2}{\left(\frac{L}{2}-b\right)[\frac{L}{2}\left(2b-\frac{L}{2}\right)]^{3/2}}.$$

Both second derivatives show:

$$\frac{\partial^2 A}{\partial b^2} + \frac{\partial^2 A}{\partial c^2},$$

the trace of the matrix to be negative, and

$\frac{\partial^2 A}{\partial b^2} \cdot \frac{\partial^2 A}{\partial c^2}$ to be positive.

Next the determinant of the matrix. We need $\frac{d^2 A}{\partial b \partial c}$, and $\frac{\partial^2 A}{\partial c \partial b}$,

$\frac{d^2 A}{\partial b \partial c} = \frac{\partial^2 A}{\partial c \partial b}$, however I calculated both

$$-\frac{\partial^2 A}{\partial c \partial b} = \frac{\partial}{\partial c}\left\{\frac{1}{2}\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)}}\left[\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-c\right)\right]\right\} =$$

$$= -\frac{1}{2}\frac{\frac{L}{2}[\left(\frac{L}{2}-b\right)+\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)]}{\sqrt{\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)}} - \frac{1}{4}\frac{(\frac{L}{2})^2[\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-c\right)][\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)]}{[\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)]^{3/2}},$$

$$-\frac{d^2 A}{\partial b \partial c} = \frac{\partial}{\partial b}\left\{\frac{1}{2}\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)}}\left[\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\right]\right\} =$$

$$= -\frac{1}{2}\frac{\frac{L}{2}[\left(\frac{L}{2}-c\right)+\left(\frac{L}{2}-b\right)-\left(b+c-\frac{L}{2}\right)]}{\sqrt{\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)}} - \frac{1}{4}\frac{(\frac{L}{2})^2[\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)][\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)-\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-c\right)]}{[\frac{L}{2}\left(b+c-\frac{L}{2}\right)\left(\frac{L}{2}-b\right)\left(\frac{L}{2}-c\right)]^{3/2}}.$$

Again, using $b = c$:

- $\frac{\partial^2 A}{\partial c \partial b} = -\frac{1}{2}\frac{\frac{L}{2}\left[\frac{3}{2}L-4b\right]}{\left(\frac{L}{2}-b\right)\sqrt{\frac{L}{2}\left(2b-\frac{L}{2}\right)}} - \frac{1}{4}\frac{(\frac{L}{2})^2[l-3b]^2}{\left(\frac{L}{2}-b\right)[\frac{L}{2}\left(2b-\frac{L}{2}\right)]^{3/2}}$,

- $\frac{\partial^2 A}{\partial b \partial c} = -\frac{1}{2}\frac{\frac{L}{2}\left[\frac{3}{2}L-4b\right]}{\left(\frac{L}{2}-b\right)\sqrt{\frac{L}{2}\left(2b-\frac{L}{2}\right)}} - \frac{1}{4}\frac{(\frac{L}{2})^2[l-3b]^2}{\left(\frac{L}{2}-b\right)[\frac{L}{2}\left(2b-\frac{L}{2}\right)]^{3/2}}$.

Consequently

$\frac{d^2 A}{\partial b \partial c} \cdot \frac{\partial^2 A}{\partial c \partial b} = (\frac{\partial^2 A}{\partial c \partial b})^2$, is positive.

Then, compare $\frac{\partial^2 A}{\partial b^2} \cdot \frac{\partial^2 A}{\partial c^2}$ with $(\frac{\partial^2 A}{\partial c \partial b})^2$.

With the above expressions it sufficient to compare

$\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}\left(2b-\frac{L}{2}\right)}}$,

the first term of $\frac{\partial^2 A}{\partial b^2}$,

and

$\frac{1}{2}\frac{\frac{L}{2}\left[\frac{3}{2}L-4b\right]}{\left(\frac{L}{2}-b\right)\sqrt{\frac{L}{2}\left(2b-\frac{L}{2}\right)}}$,

the first term of $\frac{\partial^2 A}{\partial c \partial b}$.

So, the question is

$\frac{\frac{L}{2}}{\sqrt{\frac{L}{2}\left(2b-\frac{L}{2}\right)}} > \frac{1}{2}\frac{\frac{L}{2}\left[\frac{3}{2}L-4b\right]}{\left(\frac{L}{2}-b\right)\sqrt{\frac{L}{2}\left(2b-\frac{L}{2}\right)}}$?

After some algebra, we obtain:

$L - 2b > \frac{3}{2}L - 4b$?

The perimeter $L = a + b + c = a + 2b \Longrightarrow L - 2b = a$.

Then,

$\frac{3}{2}L - 3b = \frac{3}{2}a \Longrightarrow \frac{3}{2}L - 4b = \frac{3}{2}a - b$.

So,

$L - 2b > \frac{3}{2}a - b$?

With $L = a + 2b$, the preceding inequality becomes:

$b > \frac{1}{2}a$.

For an isosceles triangle with the equal sides to be b and $c(= b)$, the preceding inequality is true.

Hence:

$\frac{\partial^2 A}{\partial b^2}\frac{\partial^2 A}{\partial c^2} - \frac{d^2 A}{\partial b \partial c}\ \frac{\partial^2 A}{\partial c \partial b} > 0.$

Consequently, the determinant of the matrix is positive and the trace negative and we have a local maximum.

There is more.

Completely similar to the foregoing analysis, with $c = L - b - a$, we obtain
$b = a$.
Hence, with $b = c$ and $b = a$, the optimal area of a triangle for a given perimeter L is the area of an equilateral triangle with sides equal to $\frac{L}{3}$.
The area for an equilateral triangle with perimeter L is: $A_t = \frac{1}{12\sqrt{3}} L^2$.
Note: we could have analysed the case $b = L - c - a$, with the result $c = a$. Obviously, no new information is created.
We did not find a circle!

Another Method
We have the formula for the area of the triangle:
$$A_t = \sqrt{\frac{L}{2}(b + c - \frac{L}{2})(\frac{L}{2} - b)(\frac{L}{2} - c)}$$
For convenience, I make this formula dimensionless with the perimeter
$$A_t' = \frac{A_t}{(\frac{L}{2})^2} = \sqrt{(b' + c' - 1)(1 - b')(1 - c')} .$$
In the following analysis I drop the primes.
$$A_t = \sqrt{(b + c - 1)(1 - b)(1 - c)} .$$
Without loss of generality make c a fraction of $b \Longrightarrow c = fb$,
where $\{f \in \mathbb{R} | f > 0\}$.
So,
$$A_t = \sqrt{(b + fb - 1)(1 - b)(1 - fb)} .$$
Now , differentiate with respect to f.
Set $\frac{dA_t}{df} = 0 \Longrightarrow b = \frac{2}{2f+1}$.
Do we find for $\frac{dA_t}{df} = 0$, a maximum?
To find out we calculate the second derivative and substitute $b = \frac{2}{2f+1}$ into $\frac{d^2A_t}{df^2}$:
$\frac{d^2A_t}{df^2} = -2(2f + 1)^{5/2}(2f - 1)^{3/2} \Longrightarrow \frac{d^2A_t}{df^2} < 0$, for $f > \frac{1}{2}$.
We have a maximum for $f > \frac{1}{2}$.
The expression for A_t with $b = \frac{2}{2f+1}$ reads
$$A_t = \frac{(2f-1)^{1/2}}{(2f+1)^{3/2}} .$$
The value of f giving the maximum value of A_t:
$\frac{dA_t}{df} = 0 \Longrightarrow \frac{3(2f-1)}{2f+1} = 1 \Longrightarrow f = 1 > \frac{1}{2}$.
Plug $f = 1$ into $A_t = \frac{(2f-1)^{1/2}}{(2f+1)^{3/2}}$:
$$A_t = \frac{1}{3\sqrt{3}} ,$$
or in the dimensions of length, the perimeter,

$A_t = \frac{1}{12\sqrt{3}} L^2$.
The familiar expression for a equilateral triangle. We obtained above.

§ 2.2 The rectangle and the Square compared with the Circle

As Luttwak mentioned the rectangular pattern, the square, became dominant over old-style rectangles in building fortresses . Well, this change is easy to understand with a little bit of calculus.
We have a rectangle with sides (a, b) and a perimeter $L = 2a + 2b$.
The area A becomes: $A = ab$.
For a given value of L we can express the area in L and a.
$A = a(\frac{L}{2} - a)$.
Optimizing A for a given L we find $a = L/4$: a square. This can be done by trial and error. However, with a little bit of calculus the optimum value for a follows from
$\frac{dA}{da} = 0 \rightarrow \frac{L}{2} - 2a = 0 \rightarrow a = \frac{L}{4}$.
Is this optimum a maximum?
$\frac{d^2A}{da^2} = -2 \rightarrow$ a maximum indicated by the – sign.
Then we have for the area A of a square : $\frac{A_s}{L^2} = \frac{1}{16}$,
where the subscript s indicates the square.
The same result is obtained starting with two isosceles triangles opposite to each other, i.e., with a common base x.
See Figure 2.1:

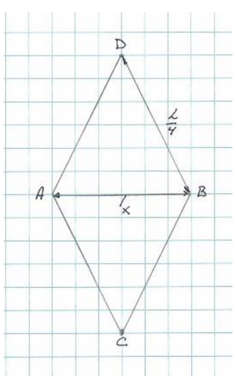

Figure 2.1 Two Isosceles Triangles, a rhomb.

The total area A of the two triangles with perimeter L each, leaving out the details,
$A = \frac{x}{4}\sqrt{L^2 - 4x^2}$.
$\frac{dA}{dx} = 0 \Longrightarrow \frac{1}{4}\sqrt{L^2 - 4x^2} - \frac{x^2}{\sqrt{L^2 - 4x^2}} = 0 \Leftrightarrow L^2 - 8x = 0 \Longrightarrow x = \frac{\sqrt{2}}{4}L$.
Then, with $x = \frac{\sqrt{2}}{4}L$

$A = (\frac{L}{4})^2 \Longrightarrow \frac{A_s}{L^2} = \frac{1}{16}$.

The area of a square. Furthermore, $\frac{d^2 A}{dx^2} < 0$, a maximum indeed.

What about the circle and the square with the same perimeter?

For the square we have $\frac{A_s}{L^2} = \frac{1}{16}$.

The area A_c of the circle with perimeter L and radius $r = \frac{L}{2\pi}$:

$A_c = \pi(\frac{L}{2\pi})^2 \Longrightarrow \frac{A_c}{L^2} = \frac{1}{4\pi}$.

Hence $\frac{A_c}{A_s} = \frac{4}{\pi} > 1$, as it should be.

§ 2.3 The Square, the Hexagon, and the Circle

The hexagon consists of six equilateral triangles. Consequently, the sides s of these triangles have a length of $\frac{L}{6}$. The perimeter of a single equilateral triangle equals $3s = \frac{L}{2}$.

The area A_h of the hexagon using $s = \frac{L}{6}$,

$A_h = \frac{3\sqrt{3}}{2} s^2 \Longrightarrow A_h = \frac{1}{8\sqrt{3}} L^2$.

So, let us compare the areas of the equilateral triangle, the square, the hexagon and the circle:

- $A_t = \frac{1}{12\sqrt{3}} L^2$
- $A_s = \frac{L^2}{16}$,
- $A_h = \frac{1}{8\sqrt{3}} L^2$,

and

- $A_c = \frac{L^2}{4\pi}$.

Hence

$A_t < A_s < A_h < A_c$, $(4\sqrt{3} < 9 < 6\sqrt{3} < 36/\pi)$,

all areas with the same perimeter L.

§ 2.4 The Square, the Hexagon, the regular Polygon, and the Circle

The polygon consist of n isosceles triangles.

The top angle β of the isosceles triangles equals

$\beta = \frac{2\pi}{n}$.

The length of the basis of these triangles is $\frac{L}{n}$.

The area $\frac{A_p}{n}$ of the isosceles triangle:

$\frac{A_p}{n} = \frac{1}{2} \frac{(\frac{L}{2n})^2}{1-\cos^2(\frac{\beta}{2})} \sin\beta$.

After some algebra, A_p is

$\frac{A_p}{n} = \frac{1}{\tan(\frac{\pi}{n})}(\frac{L}{2n})^2.$

Hence,

$A_p = \frac{1}{(\frac{n}{\pi})\tan(\frac{\pi}{n})}\frac{L^2}{4\pi},$

with $\frac{\frac{\pi}{n}}{\tan\frac{\pi}{n}} \leq 1$

For $n = 6$:

$A_p = A_h = \frac{1}{8\sqrt{3}}L^2.$

For $n \rightarrow \infty$,

$\left(\frac{n}{\pi}\right)\tan\left(\frac{\pi}{n}\right) = 1 \Longrightarrow A_p = A_c = \frac{L^2}{4\pi}$, as to be expected.

Note: the areas of the equilateral triangle, the square and the hexagon can be obtained with the expression for the polygon:

$A_p = \frac{1}{(\frac{n}{\pi})\tan(\frac{\pi}{n})}\frac{L^2}{4\pi},$

with $n = 3{,}4$, and 6 respectively.

In Figure 2.2 below I illustrate the function

$y = \frac{x}{\tan x} = \frac{1}{\frac{1}{x}\tan x},$

with WolframAlpha, x is the horizontal axis.

For the equilateral triangle: $x = \frac{\pi}{3}$,

for the square: $x = \frac{\pi}{4}$,

for the hexagon: $x = \frac{\pi}{6}$,

for the polygon: $x = \frac{\pi}{n}, n > 6$,

and

for the circle : $x = \frac{\pi}{n}, n \rightarrow \infty$.

So, for the optimal areas A_t, A_s, A_h, A_p: $0 \leq x \leq \pi/3$.

The maximum value is found for $x = 0$: the circle.

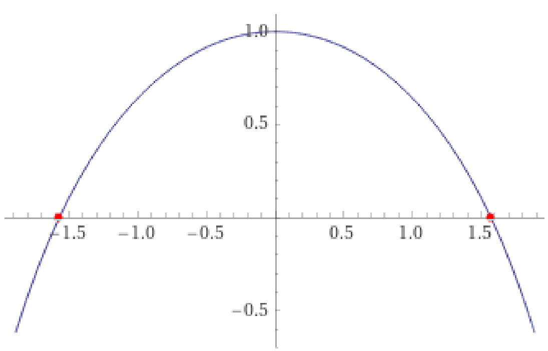

Figure 2.2 y=x/tanx as a function of x

For $y = 0$: $x = \frac{\pi}{2}$.
In the Figure 2.3 below $y = \frac{\tan x}{x}$ has been plotted with WolframAlpha.

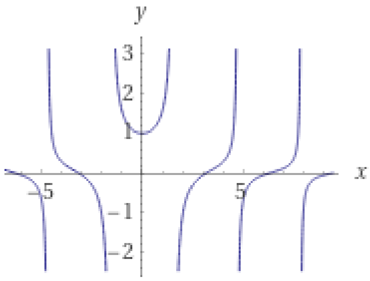

Figure 2.3 y=tanx/x as a function of x

For completeness I also present in Figure 2.4 a graph of $y = \frac{\sin x}{x}$, with Wolfram Alpha.

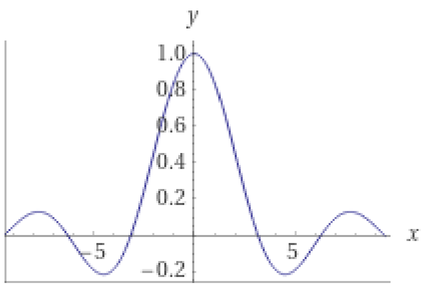

Figure 2.4 y=sinx/x as a function of x

§ 2.5 The Half Circle

To complete the analysis, I relate the area of a half circle and its perimeter L with the area of the circle A_c , with the same perimeter, : $\frac{L^2}{4\pi}$
We take the area of half a circle with radius R_0. The area A_0 of this half circle is:
$A_0 = \frac{1}{2}\pi R_0^2,$
and the perimeter is
$L = \pi R_0 + 2R_0.$
Then,

$$\frac{A_0}{L^2} = \frac{\frac{1}{2}\pi R_0^2}{(\pi R_0 + 2R_0)^2} = \frac{\pi}{2(\pi+2)^2} \Longrightarrow A_0 = \frac{L^2}{4\pi}\frac{2\pi^2}{(\pi+2)^2} = \frac{2}{\left(1+\frac{2}{\pi}\right)^2}\frac{L^2}{4\pi} \cong 0.747\frac{L^2}{4\pi}.$$

This expression shows a factor $\frac{2}{(1+\frac{2}{\pi})^2} < 1$.

The ratio A_c/A_0, both areas with the same perimeter L, is $\frac{\frac{L^2}{4\pi}}{\frac{L^2}{4\pi}\frac{2}{\left(1+\frac{2}{\pi}\right)^2}} = \frac{\left(1+\frac{2}{\pi}\right)^2}{2} > 1$.

§ 3 The Isoperimetric (In)Equality

§ 3.1 A Perturbed Circle and Fourier Transform

The area enclosed by perimeter L is A. For a given length L , we have the inequality $A \leq \frac{L^2}{4\pi}$.

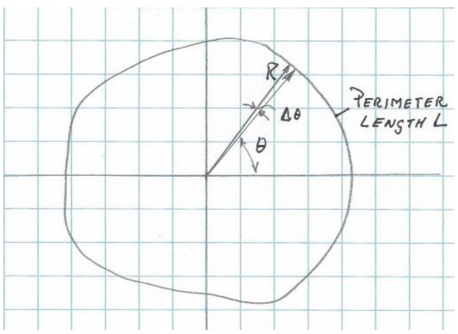

Figure 3.1 A smooth closed curve with perimeter L and area A

For further analysis I will use polar coordinates: (R, θ), Figure 3.1
I suppose the curve to be smooth and enclosing the area A to be convex. The length of the curve, the perimeter to be L.
The area of the small triangle $\frac{1}{2}R^2\Delta\theta$
Then, for $\Delta\theta \rightarrow 0$ we obtain for the area A:
$A = \int_0^{2\pi} \frac{1}{2}R^2(\theta)d\theta,$
and for the given length L of the perimeter:
$L = \int_0^{2\pi} R(\theta)d\theta.$

The Proof

I create a symmetrical curve, Figure 3.2 below, with symmetry axis's the x-axis and the y- axis. As a basis I use a circle with radius $R_1$and add to the upper half of the circle $\Delta R_1 \, sin \, \theta$; the perturbation.

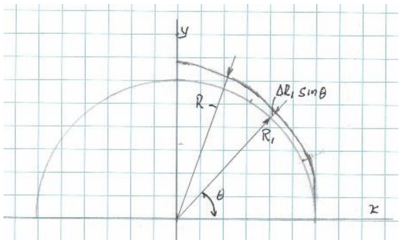

Figure 3.2 A perturbed circle

So, the radius,

$R = R_1 + \Delta R_1 \sin\theta$,

where I have chosen ΔR_1 to keep the curve described by R convex.

The area of this new circle is:

$A = 2\int_0^\pi \frac{1}{2} R^2(\theta) d\theta$.

With $R = R_1 + \Delta R_1 \sin\theta$, we find for

$A = \pi R_1^2 + 4R_1\Delta R_1 + \frac{\pi}{2}(\Delta R_1)^2$.

The given perimeter L, $L = \int_0^{2\pi} R(\theta) d\theta$, and $R = R_1 + \Delta R_1 \sin\theta$

$L = 2\int_0^\pi R(\theta) d\theta = 2\pi R_1 + 4\Delta R_1$.

Hence,

$\Delta R_1 = \frac{L - 2\pi R_1}{4}$.

We plug $\Delta R_1 = \frac{L-2\pi R_1}{4}$ into

$A = \pi R_1^2 + 4R_1\Delta R_1 + \frac{\pi}{2}(\Delta R_1)^2$:

$A = \pi R_1^2 + R_1 L - 2\pi R_1^2 + \frac{\pi}{32}(L^2 - 4\pi L R_1 + 4\pi^2 R_1^2)$.

Next we optimize A:

$\frac{dA}{dR_1} = 0 = \frac{d}{dR_1}\left[\pi R_1^2 + R_1 L - 2\pi R_1^2 + \frac{\pi}{32}(L^2 - 4\pi L R_1 + 4\pi^2 R_1^2)\right] \Longrightarrow R_1 = \frac{L}{2\pi} \Longrightarrow \pi R_1^2 = \frac{L^2}{4\pi}$.

Note: R_1 is the radius of the optimized are A, a circle of which the perimeter

$L = 2\pi R_1 + 4\Delta R_1$.

In the preceding expression for the perimeter $R_1$is the radius of the "old" circle.

Caveat convex!

The question to be answered: is the curve presented in the above Figure 3.2 convex? Alas, it is not.

With $R = R_1 + \Delta R_1 \sin\theta$, we have

$\frac{dR}{d\theta} = \Delta R_1 \cos\theta$.

Consequently, $\theta \to 0 \Longrightarrow \frac{dR}{d\theta} = \Delta R_1 > 0$. The curve is concave.

How to deal with that? There are two possibilities to make the curve convex.

I choose the most simple one and create a new area by $R = R_1 - \Delta R_1 \sin\theta$.

Now,

$\Delta R_1 = \frac{2\pi R_1 - L}{4}$.

After some algebra :

$\frac{dA}{dR_1} = 0 = \frac{d}{dR_1}\left[\pi R_1^2 + R_1 L - 2\pi R_1^2 + \frac{\pi}{32}(L^2 - 4\pi L R_1 + 4\pi^2 R_1^2)\right] \Longrightarrow R_1 = \frac{L}{2\pi} \Longrightarrow \pi R_1^2 = \frac{L^2}{4\pi}$.

To prevent confusion, the radius resulting from the preceding expression is denoted R_n, and

$\pi R_n^2 = \frac{L^2}{4\pi}$, *and*

$L = 2\pi R_1 - 4\Delta R_1$.

A maximum for $R_1 = \frac{L}{2\pi}$?

$\frac{d^2 A}{dR_1{}^2} = -2\pi(1 - \frac{\pi^2}{16}) < 0 \Longrightarrow$ *a maximum.*

End of Proof

§ 3.2 A Convex Area Composited of n different Isosceles Triangles with the Same Top Angle

Another approach to prove the isoperimetric inequality. Or the maximum area for a given perimeter.

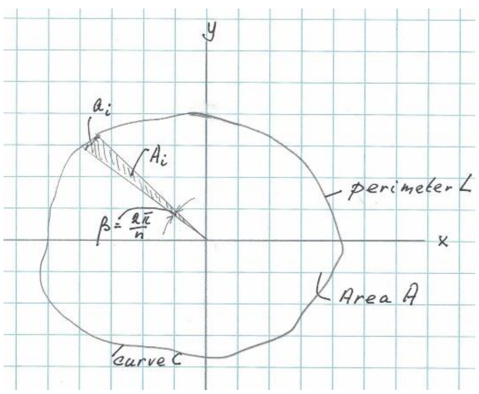

Figure 3.3 Area A composited of n, different, isosceles triangles with top angle 2π/n, basis a_i and area A_i.$i = 1(1)n$.

Another Proof

The area A is constructed by n isosceles triangles. The top of the triangles is somewhere in the middle of the area A, the origin of the cartesian frame. The triangles have all the same top angle $\beta = 2\pi/n$. The area A_i of triangle i with base a_i:

$A_i = \frac{1}{tan\frac{\pi}{n}}(\frac{a_i}{2})^2$.

Furthermore, the area A is contained by curve C, with length L (the given perimeter)

$L = \sum_{i=1}^{n} a_i$,

where the curve segment is approximated by a_i .

The area A:

$A = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{1}{tan\frac{\pi}{n}}(\frac{a_i}{2})^2 = \frac{\pi}{tan\frac{\pi}{n}}\sum_{i=1}^{n}\frac{(a_i)^2}{4\pi}$.

Now, the question is: can we proof for a given perimeter L the area of the circle to be the maximum possible area?

I optimize the area $A = \frac{\pi}{tan\frac{\pi}{n}} \sum_{i=1}^{n} \frac{(a_i)^2}{4\pi}$,
and optimize for the variable a_i*:*
$\frac{dA}{da_k} = 0$.
I use the perimeter $L \rightarrow a_{k+1} = L - (\sum_{i=1}^{k} a_i + \sum_{i=k+2}^{n} a_i)$*:*
$\frac{d}{da_k}\{\sum_{i=1}^{k-1}(a_i)^2 + (a_k)^2 + \left(L - \left[\sum_{i=1}^{k-1} a_i + a_k + \sum_{i=k+2}^{n} a_i\right]\right)^2 + \sum_{i=k+2}^{n}(a_i)^2\} = 0 \rightarrow$
$\rightarrow 2a_k + 2\left(L - \left[\sum_{i=1}^{k} a_i + \sum_{i=k+2}^{n} a_i\right]\right) \cdot (-1) = 0 \rightarrow 2a_k - 2a_{k+1} = 0 \Longrightarrow a_k = a_{k+1}$,
for $1 \leq k \leq n - 1$,
and $L - \left[\sum_{i=1}^{k} a_i + \sum_{i=k+2}^{n} a_i\right] = a_{k+1}$ *has been plugged back into the expression for the derivative.*
Hence, the optimization process leads to n *equal isosceles triangles. Consequently, the optimized area* $A = A_{max} = A_p$ *is the area of a n-polygon and* $a_i = \frac{L}{n}$*, § 2.4, with perimeter* L*.*
The curve C, Figure 3.3, is changed in such a way the area of n-polygon of equal isosceles triangles is the maximum area. What about the circle?
Well,
$A = A_p = \frac{\pi}{tan\frac{\pi}{n}} \sum_{i=1}^{n} \frac{(L/n)^2}{4\pi} = \frac{\frac{\pi}{n}}{tan\frac{\pi}{n}} \frac{L^2}{4\pi}$,
where $\frac{\frac{\pi}{n}}{tan\frac{\pi}{n}} \leq 1$*, for* $0 \leq \frac{\pi}{n} \leq \frac{\pi}{2}$.
Then, see § 2.4, and $\lim_{n\to\infty} \frac{\frac{\pi}{n}}{tan\frac{\pi}{n}} = 1$,
$\lim_{n\to\infty} A_{max} = \frac{L^2}{4\pi}$,
the isoperimetric equality. The maximum area is the area of a circle.

Note : I have chosen a_{k+1}*. This choice is not special. I could have chosen* a_{k+4} *for that matter.*
I addressed the problem of finding the maximum area A for a given perimeter of length L in a cartesian frame. The circle creates the largest area.
End of Proof
Caveat: Is this proof complete? Well, as you notice $\frac{d^2A}{da_k{}^2} = 4$. Hence, do we have a maximum or a minimum? So, alas this proof is not complete. We do have more than two independent variables. Consequently, the rules to find out about a maximum or a minimum are more complicated.

§ 3.3 The Optimization of three Isosceles Triangles

$A = \sum_{i=1}^{3} A_i = \sum_{i=1}^{3} \frac{1}{\tan\frac{\pi}{3}} (\frac{a_i}{2})^2 = \frac{1}{4\tan\frac{\pi}{3}} [a_1^2 + a_2^2 + (L - a_1 - a_2)^2]$,
with $a_3 = L - a_1 - a_2$.
By defining the perimeter L to be $L = a_1 + a_2 + a_3$,

I assumed $|a_1 - a_3|/a_3 \ll 1$ and $|a_2 - a_3|/a_3 \ll 1$.
Then

$$\frac{\partial A}{\partial a_1} = \frac{1}{4\tan\frac{\pi}{3}}[2a_1 - 2(L - a_1 - a_2)] = \frac{1}{4\tan\frac{\pi}{3}}(4a_1 + 2a_2 - 2L).$$

$$\frac{\partial A}{\partial a_2} = \frac{1}{4\tan\frac{\pi}{3}}[2a_2 - 2(L - a_1 - a_2)] = \frac{1}{4\tan\frac{\pi}{3}}(4a_2 + 2a_1 - 2L).$$

$\frac{\partial A}{\partial a_1} = \frac{\partial A}{\partial a_2} = 0$:

- $4a_1 + 2a_2 - 2L = 0$,

and

- $4a_2 + 2a_1 - 2L = 0$.

Hence

$$a_1 = a_2 = \frac{L}{3} \Longrightarrow a_3 = \frac{L}{3}.$$

Note: this is the inevitable consequence of $a_3 = L - a_1 - a_2$, and the angle $\frac{\pi}{3}$.

With the assumption $|a_1 - a_3|/a_3 \ll 1$ and $|a_2 - a_3|/a_3 \ll 1$,
$a_1 = a_3$, and $a_2 = a_3$.
Hence, there is nothing left to optimize. From the start, using the above assumptions, I assumed three **equal** isosceles triangles.
This is the reason why I will analyse in the following the $n = 3$ problem for
$(a_1 - a_3)/a_3 = O(1)$, and $(a_2 - a_3)/a_3 = O(1)$.
In Figure 3.4 below, I illustrated one of the cases to be analysed, with $(a_1 - a_3)/a_3 = O(1)$, and $(a_2 - a_3)/a_3 = O(1)$.
Furthermore, I set $a_1 = a_2$. Just for convenience. The isosceles triangles have the same top angle $\beta = 2\pi/3$.

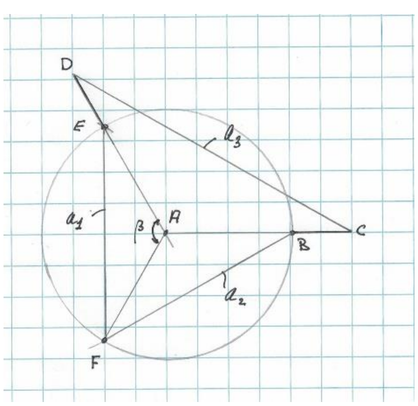

Figure 3.4 n=3 Isosceles Triangles

First I establish the perimeter L:

$L = 2a_1 + a_3 + BC + DE = 2a_1 + a_3 + 2(a_3 - a_1)/(2\ \sin\frac{\beta}{2}) = a_1(2 - \frac{2}{\sqrt{3}}) + a_3(1 + \frac{2}{\sqrt{3}})$.
Here I assume $(a_3 - a_1) \geq 0$.

> Note:
> $L = 2a_1 + a_3 + 2(a_3 - a_1)/(2\ \sin\frac{\beta}{2})$.
> This expression leads to the conclusion the maximum value of $2a_1 + a_3 = L$, for $a_3 = a_1$.
> Keep in mind: in $2a_1 + a_3 = L$, a_i differs from a_i in $L = 2a_1 + a_3 + 2(a_3 - a_1)/(2\ \sin\frac{\beta}{2})$.
> So, denote a_i in $2a_1 + a_3 = L$, a.
> Then, the maximum value of $a = \frac{L}{3}$, and
> $3a = L = 2a_1 + a_3 + 2(a_3 - a_1)/(2\ \sin\frac{\beta}{2})$.
> The proof, hidden in plain sight.
> The maximum area becomes
> $A = 3\frac{1}{4\tan\frac{\pi}{3}}a^2 = \frac{L^2}{12\sqrt{3}}$.

In the preceding note I presented the maximum area with a geometrical approach.
However, for completeness I shall analyse the expression for the area.
The area of the three isosceles triangles:
$A = 2\frac{1}{4\tan\frac{\pi}{3}}a_1^2 + \frac{1}{4\tan\frac{\pi}{3}}a_3^2$.
With the perimeter $L = a_1(2 - \frac{2}{\sqrt{3}}) + a_3(1 + \frac{2}{\sqrt{3}})$:
$a_3 = \frac{1}{(1+\frac{2}{\sqrt{3}})}\left[L - a_1\left(2 - \frac{2}{\sqrt{3}}\right)\right] = \frac{2}{2+\sqrt{3}}[\frac{\sqrt{3}}{2}L - a_1(\sqrt{3} - 1)]$,
and
$a_1(\sqrt{3} - 1) < \frac{\sqrt{3}}{2}L \Longrightarrow \frac{a_1}{L} < \frac{3+\sqrt{3}}{4}$.
Remark: $\frac{a_1}{L} < \frac{3+\sqrt{3}}{4}$ is a trivial constraint , since $\frac{a_1}{L} < 1$.
We can improve this constraint with the expression for the perimeter and $(a_3 - a_1) \geq 0$:
$L = 2a_1 + a_3 + 2(a_3 - a_1)/(2\ \sin\frac{\beta}{2})$.
With $(a_3 - a_1) \geq 0$, the smallest possible value of a_3 is a_1.
Hence, $\frac{a_1}{L} \leq \frac{1}{3}$.
For $a_1 = 0 \Longrightarrow \frac{a_3}{L} = (2\sqrt{3} - 3)$.
The latter result is obtained for one isosceles triangle with perimeter L and base a_3 and top angle $\beta = \frac{2\pi}{3}$.

$a_3 \Longrightarrow 0$, is a contradiction, since $(a_3 - a_1) \geq 0$ and a given value of the perimeter.

By the way, what do we find for $\frac{a_1}{L}$, choosing $\frac{a_3}{L} = \frac{1}{3}$?
Well, using the above expressions for $\frac{a_3}{L}$:
- $a_3 = \frac{2}{2+\sqrt{3}}[\frac{\sqrt{3}}{2}L - a_1(\sqrt{3}-1)]$,
or
- $L = a_1(2 - \frac{2}{\sqrt{3}}) + a_3(1 + \frac{2}{\sqrt{3}})$,
the result is: $\frac{a_1}{L} = \frac{1}{3}$.

For the area $A = 2\frac{1}{4\tan\frac{\pi}{3}}a_1^2 + \frac{1}{4\tan\frac{\pi}{3}}a_3^2$, and $a_3 = \frac{2}{2+\sqrt{3}}[\frac{\sqrt{3}}{2}L - a_1(\sqrt{3}-1)]$:

$$A = \frac{1}{2\sqrt{3}}a_1^2 + \frac{1}{4\sqrt{3}}\frac{4}{7+4\sqrt{3}}\left[\frac{\sqrt{3}}{2}L - a_1(\sqrt{3}-1)\right]^2,$$

or in dimensionless form with $\frac{a_1}{L} \equiv x$,

$$A' = \frac{A}{L^2} = \frac{1}{2\sqrt{3}}x^2 + \frac{1}{7\sqrt{3}+12}\left[\frac{\sqrt{3}}{2} - x(\sqrt{3}-1)\right]^2$$

a parabolic expression.

Next

$$\frac{dA'}{dx} = \frac{x}{\sqrt{3}} - \frac{2}{7\sqrt{3}+12}\left[\frac{\sqrt{3}}{2} - x(\sqrt{3}-1)\right](\sqrt{3}-1).$$

With the parabolic equation for A or A' we can find the minimum value, $\frac{d^2A'}{dx^2} > 0$, for the area: $\frac{dA'}{dx} = 0$.

This minimum value is found for $\frac{a_1}{L} = x = \frac{3-\sqrt{3}}{15} \Longrightarrow \frac{a_3}{L} = \frac{3+2\sqrt{3}}{15}$.

So the minimum area, at $x \cong 0.085$, is $\frac{3+13\sqrt{3}}{900} \cong 0.028$.

In Figure 3.5 below $A'(= y)$ is represented as a function of x .

The area $A' = \frac{7\sqrt{3}-12}{4}(\cong 0.031)$, for $x = 0$.

The upper value of $\frac{a_1}{L} = \frac{1}{3}$, indicates this to be the local maximum value for A'. Let us find out. Since $\frac{a_1}{L} = \frac{1}{3}$, $a_3 = a_1$, we need to find the area of three equal isosceles triangles.[2]

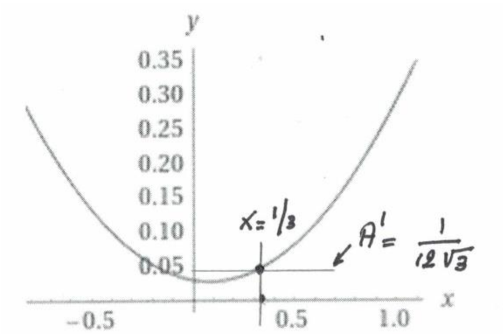

Figure 3.5 Parabola representing the dimensionless area of A'.

We use the expression:

$$A = 2\frac{1}{4\tan\frac{\pi}{3}}a_1^2 + \frac{1}{4\tan\frac{\pi}{3}}a_3^2 \Longrightarrow A' = \frac{3}{4\sqrt{3}}\left(\frac{a_1}{L}\right)^2 \Leftrightarrow A' = \frac{1}{12\sqrt{3}} \cong 0.048, \text{ at } x = \frac{1}{3}.$$

The other local maximum value for A', at $x = 0$: $A' = \frac{7\sqrt{3}-12}{4} \cong 0.031 < \frac{1}{12\sqrt{3}} \cong 0.048.$

Hence, as we found earlier, the maximum area for the given perimeter L is obtained for $x = \frac{1}{3}$: *three equal isosceles triangles.*

Note: the sum of areas of the three isosceles triangles equals the area of a equilateral triangle with sides $a_1 = a_2 = a_3$: $A' = \frac{1}{12\sqrt{3}}$, § 2.1 .

The preceding analysis can be illustrated with the geometrical picture in Figure 5.

Let us have a closer look at Figure 3.6.

With $L = 2a_1 + a_3 + BC + DE$, we can contain a larger area than the sum of the presented isosceles triangles. This is demonstrated in Figure 3.6 below.

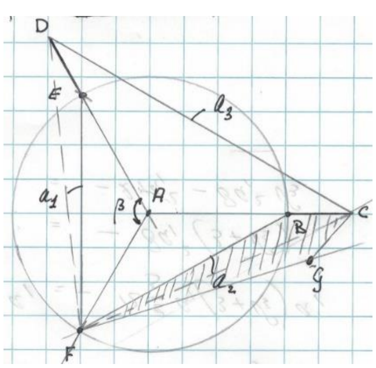

Figure 3.6 Shaded area demonstrates the larger area for the same value of L, with $a_3 - a_1 > 0$.

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[2] $a_3 = \frac{2}{2+\sqrt{3}}\left[\frac{\sqrt{3}}{2}L - a_1(\sqrt{3}-1)\right]$. With $\frac{a_1}{L} = \frac{1}{3} \Longrightarrow a_3 = a_1$.

By connecting the points F and C , or DF for that matter, and mirror the ΔFBC with respect to the line FC, we obtain the shaded area. The perimeter L applies to A, as defined above plus twice the shaded area.
Hence, there is a maximum area.
Keep in mind the isosceles triangles remain isosceles triangles. Consequently, new values for a_1,a_2 and a_3 are found. The resulting configuration needs to be analysed whether there still exist concave parts. If so, we repeat the construction to get rid of the concave part of the perimeter L. As you notice, for this case in Figure 3.6, there are no concave parts left.
The result of this process are three equal isosceles triangles with $a_1 = a_2 = a_3$.

I started the discussion on the subject matter with the assumption $(a_3 - a_1) \geq 0$, leading to the maximum area for $\frac{a_1}{L} = \frac{1}{3}$.
Let us analyse this assumption. To formulate the perimeter in a correct way, Figure 3.4, the result is:

$$L = 2a_1 + a_3 + BC + DE = 2a_1 + a_3 + 2|a_3 - a_1| \frac{1}{2\ \sin\frac{\beta}{2}} \Longrightarrow$$

$$\Longrightarrow |a_3 - a_1| = \frac{\sqrt{3}}{2}(L - 2a_1 - a_3) \Longrightarrow a_3 - a_1 = \pm\frac{\sqrt{3}}{2}(L - 2a_1 - a_3).$$

***Note**: Since $|a_3 - a_1| \geq 0$, the maximum possible value of $2a_1 + a_3 = L$.* *So with $|a_3 - a_1| = 0 \Longrightarrow a_1 = a_3 = \frac{1}{3} L$.* *Hence the inevitable conclusion: the maximum possible area for a given perimeter L results from the sum of three equal isosceles triangles.*

In the foregoing analysis I investigated

$$a_3 - a_1 = +\frac{\sqrt{3}}{2}(L - 2a_1 - a_3),$$

with $L \geq 2a_1 + a_3$.
Consequently, $\frac{a_1}{L} \leq \frac{1}{3}$.

Next, we analyse, with $L \geq 2a_1 + a_3$:

$$a_3 - a_1 = -\frac{\sqrt{3}}{2}(L - 2a_1 - a_3) \Longrightarrow a_3 = -\frac{2}{2-\sqrt{3}}\left[\frac{\sqrt{3}}{2}L - a_1\left(1 + \sqrt{3}\right)\right],$$

and , with $a_3 \geq 0$

$$a_1\left(1 + \sqrt{3}\right) \geq \frac{\sqrt{3}}{2}L \Longrightarrow \frac{a_1}{L} \geq \frac{3-\sqrt{3}}{4}.$$

The range for $\frac{a_1}{L}$, with $(a_3 - a_1) \leq 0$,

$$\frac{3-\sqrt{3}}{4} \leq \frac{a_1}{L} \leq \frac{1}{3}.$$

The upper bound for $\frac{a_1}{L}$ is again $\frac{1}{3}$.

The area of the three isosceles triangles, with the preceding expression for a_3,

$$A = 2\frac{1}{4\tan\frac{\pi}{3}}a_1^2 + \frac{1}{4\tan\frac{\pi}{3}}a_3^2 = \frac{1}{2\sqrt{3}}\left(a_1^2 + \frac{1}{2}a_3^2\right) = \frac{a_1^2}{2\sqrt{3}} + \frac{1}{7\sqrt{3}-12}\left[-\frac{\sqrt{3}}{2}L + a_1\left(1+\sqrt{3}\right)\right]^2.$$

Or dimensionless:

$$A' = \frac{A}{L^2} = \frac{1}{2\sqrt{3}}x^2 + \frac{1}{7\sqrt{3}-12}\left[-\frac{\sqrt{3}}{2} + x\left(1+\sqrt{3}\right)\right]^2.$$

$$\frac{dA'}{dx} = \frac{x}{\sqrt{3}} + \frac{2}{7\sqrt{3}-12}\left[-\frac{\sqrt{3}}{2} + x\left(1+\sqrt{3}\right)\right]\left(1+\sqrt{3}\right).$$

$\frac{dA'}{dx} = 0 \Longrightarrow x = \frac{3+\sqrt{3}}{15}$, and $\frac{a_3}{L} = \frac{3-2\sqrt{3}}{15} < 0(!)$, outside the range of a meaningful value of x:
$\frac{3-\sqrt{3}}{4} > \frac{3+\sqrt{3}}{15}$.

$\frac{d^2A}{da_1^2} > 0$: a minimum.

The value of x for the minimum value of A' lies outside the range:
$\frac{3-\sqrt{3}}{4} \leq x \leq \frac{1}{3}$.

Next, I plug into the parabolic expression of the area the upper value, $\frac{1}{3}$, and the lower, $\frac{3-\sqrt{3}}{4}$, value of x.

- $x = \frac{1}{3}$, three equal isosceles triangles,

$$A = 2\frac{1}{4\tan\frac{\pi}{3}}a_1^2 + \frac{1}{4\tan\frac{\pi}{3}}a_3^2 \Longrightarrow A' = \frac{3}{4\sqrt{3}}\left(\frac{a_1}{L}\right)^2 \Leftrightarrow A' = \frac{1}{12\sqrt{3}} \cong 0.048,$$

- $x = \frac{3-\sqrt{3}}{4}$, I use the parabolic expression

$$A' = \frac{A}{L^2} = \frac{1}{2\sqrt{3}}x^2 + \frac{1}{7\sqrt{3}-12}\left[-\frac{\sqrt{3}}{2} + x\left(1+\sqrt{3}\right)\right]^2 \Longrightarrow A' = \frac{2\sqrt{3}-3}{16} \cong 0.029.$$

So, we conclude the maximum value again to be found for $x = \frac{1}{3}$, a local maximum.
For this case $\frac{a_1}{L} = x = 0$ is irrelevant since $\frac{a_3}{L} < 0$, is irrelevant.
The maximum value of $A' = \frac{A}{L^2}$, can be illustrated with the geometrical picture in Figure 3.7.

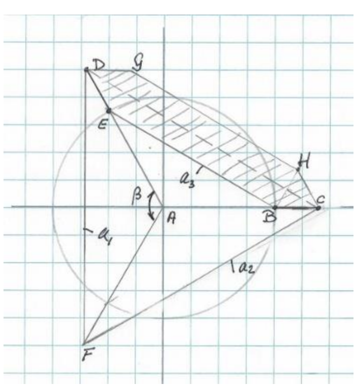


Figure 3.7 Shaded area demonstrates the larger area for the same value of L, with $a_3 - a_1 < 0$.

Here again, with $L = 2a_1 + a_3 + BC + DE$, we can contain a larger area than the sum of the presented isosceles triangles.

By connecting the points D and C and mirror the trapezium $EBCD$ with respect to the line DC, we obtain the shaded area. The perimeter L applies to A, as defined above plus the shaded area.
Hence, there is a maximum area.
Keep in mind the isosceles triangles remain isosceles triangles. Consequently, new values for a_1,a_2 and a_3 need to be found. The resulting configuration needs to be analysed whether there still exist concave parts. If so, we repeat the construction to get rid of the concave part of the perimeter L. As you notice, for this case in Figure 3.7, there are no concave parts left. The result of this process are three equal isosceles triangles with $a_1 = a_2 = a_3 = \frac{1}{3}L$.
The maximum area is represented by three equal isosceles triangles.

In the above analysis I used for convenience $a_1 = a_2$. Is it necessary?
To find out, I investigate $a_1 \neq a_2 \neq a_3$.
This case is illustrated in Figure 3.8 below.

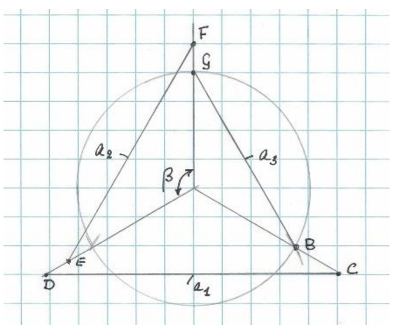


Figure 3.8 n=3, three different Isosceles triangles

The perimeter
$$L = a_1 + a_2 + a_3 + BC + DE + FG = a_1 + a_2 + a_3 + \frac{a_1 - a_3}{2\sin\frac{\beta}{2}} + \frac{a_1 - a_2}{2\sin\frac{\beta}{2}} + \frac{a_2 - a_3}{2\sin\frac{\beta}{2}}.$$
The preceding expression leads to the conclusion $a_1 + a_2 + a_3 = L$ to be the maximum value for $a_1 - a_3 = 0, a_1 - a_2 = 0$, and $a_2 - a_3 = 0$.
Then,
$$a_1 = a_2 = a_3 = \frac{1}{3}L.$$
Keep in mind:
a_i in $a_1 = a_2 = a_3 = \frac{1}{3}L$ differs from a_i in $L = a_1 + a_2 + a_3 + \frac{a_1 - a_3}{2\sin\frac{\beta}{2}} + \frac{a_1 - a_2}{2\sin\frac{\beta}{2}} + \frac{a_2 - a_3}{2\sin\frac{\beta}{2}}$.
So, to emphasize this, set a_i in $a_1 = a_2 = a_3 = \frac{1}{3}L$, equal to a, say.
Hence,
$$3 \cdot a = L = a_1 + a_2 + a_3 + \frac{a_1 - a_3}{2\sin\frac{\beta}{2}} + \frac{a_1 - a_2}{2\sin\frac{\beta}{2}} + \frac{a_2 - a_3}{2\sin\frac{\beta}{2}} = a_1 + a_2 + a_3 + \frac{2}{3}\sqrt{3}(a_1 - a_3).$$
The maximum area, for a given L, is represented by three equal isosceles triangles. As mentioned before: hidden in plain sight.

In § 3.2 an area bounded by curve C with perimeter L is approximated by n different isosceles triangles with top angles $\frac{2\pi}{n}$. The basis of triangle i is denoted a_i and the area of the triangle is A_i, with $i = 1(1)n$.
In § 3.2 we found the maximum area with the curve C can be approximated by n equal isosceles triangles. We could not prove this to be the maximum. With the preceding analysis it can be done.
The perimeter
$L = a_1 + \ a_2 + \cdots + a_i + \cdots a_{n-1} + a_n + \frac{|a_1 - a_\mathrm{n}|}{2\ \sin\frac{\beta}{2}} + \frac{|a_1 - a_2|}{2\ \sin\frac{\beta}{2}} + \cdots \frac{|a_\mathrm{i} - a_{\mathrm{i}+1}|}{2\ \sin\frac{\beta}{2}} + \cdots \frac{|a_{n-1} - a_\mathrm{n}|}{2\ \sin\frac{\beta}{2}}$,
with $\beta = \frac{2\pi}{n}$.
Hence
$a_1 + \ a_2 + \cdots + a_i + \cdots a_{n-1} + a_n = L$, is the maximum value for $|a_1 - a_\mathrm{n}| = 0$,
$|a_1 - a_2| = 0, \ldots\ldots, |a_\mathrm{i} - a_{\mathrm{i}+1}| = 0$, and $|a_{n-1} - a_\mathrm{n}| = 0$.
n equal isosceles triangles.

Keep in mind:
in $a_1 + \ a_2 + \cdots + a_i + \cdots a_{n-1} + a_n = L$, a_i differs from the a_i's in
$L = a_1 + \ a_2 + \cdots + a_i + \cdots a_{n-1} + a_n + \frac{|a_1 - a_\mathrm{n}|}{2\ \sin\frac{\beta}{2}} + \frac{|a_1 - a_2|}{2\ \sin\frac{\beta}{2}} + \cdots \frac{|a_\mathrm{i} - a_{\mathrm{i}+1}|}{2\ \sin\frac{\beta}{2}} + \cdots \frac{|a_{n-1} - a_\mathrm{n}|}{2\ \sin\frac{\beta}{2}}$.
For this reason, I denote a_i in $a_1 + \ a_2 + \cdots + a_i + \cdots a_{n-1} + a_n = L$, a. Since $|a_1 - a_\mathrm{n}| = 0$,
$|a_1 - a_2| = 0, \ldots\ldots, |a_\mathrm{i} - a_{\mathrm{i}+1}| = 0$, and $|a_{n-1} - a_\mathrm{n}| = 0$.
Hence,
$n \cdot a = L = a_1 + \ a_2 + \cdots + a_i + \cdots a_{n-1} + a_n + \frac{|a_1 - a_\mathrm{n}|}{2\ \sin\frac{\beta}{2}} + \frac{|a_1 - a_2|}{2\ \sin\frac{\beta}{2}} + \cdots \frac{|a_\mathrm{i} - a_{\mathrm{i}+1}|}{2\ \sin\frac{\beta}{2}} + \cdots \frac{|a_{n-1} - a_\mathrm{n}|}{2\ \sin\frac{\beta}{2}}$.

The area:
$A = \frac{\pi}{tan\frac{\pi}{n}} \sum_{i=1}^{n} \frac{(a)^2}{4\pi}$.
With $a = \frac{L}{n}$,
$A = \frac{\pi}{tan\frac{\pi}{n}} \sum_{i=1}^{n} \frac{(\frac{L}{n})^2}{4\pi} = \frac{\frac{\pi}{n}}{tan\frac{\pi}{n}} \frac{L^2}{4\pi}$.
The maximum area is represented by n equal isosceles triangles.

So, the area A is contained by a curve C with perimeter L. The curve C, fixed L, adapts to the maximum area of n isosceles triangles. The maximum is found for n equal isosceles triangles. The curve C consists of the sum of the equal bases of the triangles.

For $n \rightarrow \infty$, the n equal isosceles triangles represent the circle with perimeter $L \Longrightarrow$ the isoperimetric equality.

Remark: The case for $n = 3$ can also be extended for $n > 3$. For example, $n = 4$, by choosing the basis of the isosceles triangles $a_1 = a_2, \ a_3 = a_4$, and $a_3 \neq a_1$. Then, again the maximum area is represented by four equal isosceles triangles. In this way we continue with $n > 4$.

§ 4 The Isoepifaeic Inequality, some Examples

Conjecture:

The inequality: for a given surface S, the sphere to have the biggest capacity or volume V.

So, the inequality reads

$V \leq \frac{1}{6\sqrt{\pi}} S^{3/2}$.

In this paragraph we examine various geometrical structures and compare the volume of these structures with the volume of the sphere for a given surface.

§ 4.1 The Isoepifaeic Inequality of a Block with a given Surface

In Figure 2.1 below, the dimensions of the block are given.

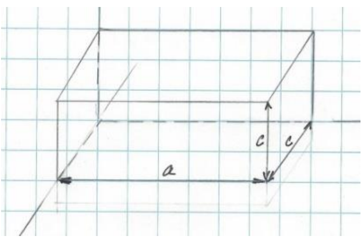


Figure 4.1 A Block wit Surface S

The surface

$S = 2c^2 + 4ac$.

Then

$a = \frac{S-2c^2}{4c}$.

$V = a \cdot c^2 = \frac{S \cdot c - 2c^3}{4}$.

Now,

$\frac{dV}{dc} = 0 = S - 6c^2 \Longrightarrow c^2 = \frac{S}{6}$, $\Longrightarrow$ the representation of a cube.

For completeness:

$a = \frac{S-2c^2}{4c} = \frac{\frac{2}{3}S}{4 \cdot \sqrt{\frac{S}{6}}} = \sqrt{\frac{S}{6}}$,

and

$\frac{d^2V}{dc^2} = -3c < 0$, a maximum.

Volume of a sphere with surface S:

$V_{sphere} = \frac{1}{6\sqrt{\pi}} S^{3/2}$,

and

$V_{cube} = (\frac{1}{6})^{3/2} S^{3/2}$.

Hence,

$\frac{V_{sphere}}{V_{cube}} = \sqrt{\frac{6}{\pi}} \cong 1.38 \Longrightarrow$ The Isoepifaeic Inequality.

§ 4.2 The Isoepifaeic Inequality of a Cone with a given Surface

In Figure 2.3 below an example of the cone is presented.
The conical surface and the surface of the ground circle included is
$S = \pi r R + \pi r^2$,

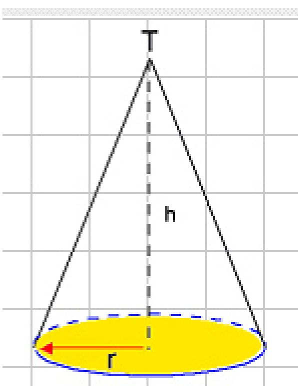


Figure 4.2 The Cone

where the slant height of the cone $R = \sqrt{r^2 + h^2} = \frac{S}{\pi r} - r$.
The volume of the cone is

$$V_{cone} = \frac{\pi}{3}\sqrt{R^2 - r^2} \cdot r^2 = \frac{\pi}{3}[\left(\frac{S}{\pi r} - r\right)^2 - r^2]^{1/2} r^2 = \frac{\pi r^2}{3}[\left(\frac{S}{\pi r}\right)^2 - 2\frac{S}{\pi}]^{1/2}.$$

Next, for a given S,

$$\frac{dV_{cone}}{dr} = 0 \Longrightarrow \frac{2\pi r}{3}\left[\left(\frac{S}{\pi r}\right)^2 - 2\frac{S}{\pi}\right]^{\frac{1}{2}} - \frac{\pi r^2}{3}\frac{\frac{S^2}{\pi^2 r^3}}{\left[\left(\frac{S}{\pi r}\right)^2 - 2\frac{S}{\pi}\right]^{\frac{1}{2}}} = 0 \Longrightarrow \left(\frac{S}{\pi r}\right)^2 - 2\frac{S}{\pi} - \frac{r}{2}\frac{S^2}{\pi^2 r^3} = 0 \Longrightarrow$$

$$\Longrightarrow \left(\frac{S}{\pi r}\right)^2 - 2\frac{S}{\pi} - \frac{1}{2}\left(\frac{S}{\pi r}\right)^2 = 0 \Longrightarrow r = \frac{1}{2} \cdot \sqrt{\frac{S}{\pi}} \Longrightarrow R = \frac{3}{2} \cdot \sqrt{\frac{S}{\pi}}.$$

What about $\frac{d^2 V_{cone}}{dr^2}$?

I leave out the details, after some algebra, $\frac{d^2 V_{cone}}{dr^2} < 0$.

Hence for

$r = \frac{1}{2} \cdot \sqrt{\frac{S}{\pi}} \Longrightarrow R = \frac{3}{2} \cdot \sqrt{\frac{S}{\pi}}$, we find a maximum volume of the cone for a given surface of the cone.

The ratio of the volume of the sphere to the volume of the cone with a given surface S and

$r = \frac{1}{2} \cdot \sqrt{\frac{S}{\pi}} \Longrightarrow R = \frac{3}{2} \cdot \sqrt{\frac{S}{\pi}}$,

$\frac{V_{sphere}}{V_{cone}} = \frac{\frac{1}{6\sqrt{\pi}} S^{3/2}}{\frac{\pi}{3}\sqrt{R^2 - r^2} \cdot r^2} = \frac{\frac{\pi}{6}(\frac{S}{\pi})^{3/2}}{\frac{\pi\sqrt{2}}{12}(\frac{S}{\pi})^{3/2}} = \sqrt{2} \Longrightarrow$ the Isoepifaeic Inequality for the cone.

§ 4.3 The Isoepifaeic Inequality for a Cylinder with a given Surface

The surface, S, of a cylinder with length L and radius r:

$S = 2\pi rL + 2\pi r^2$.

So

$L = \frac{S}{2\pi r} - r$.

The volume, V_{cyl}, of the cylinder

$V_{cyl} = \pi r^2 L = \frac{1}{2} S \cdot r - \pi r^3$.

Then

$\frac{dV_{cyl}}{dr} = 0 = \frac{1}{2} S - 3\pi r^2 \Longleftrightarrow r = \frac{1}{\sqrt{6}} \cdot \sqrt{\frac{S}{\pi}} \Longrightarrow L = \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{S}{\pi}}$.

$\frac{d^2 V_{cyl}}{dr^2} = -6\pi r < 0 \Longrightarrow$ a maximum.

Hence,

for $r = \frac{1}{\sqrt{6}} \sqrt{\frac{S}{\pi}} \Longrightarrow L = \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{S}{\pi}}$, we obtain a maximum of the volume of the cylinder for a given surface S.

The maximum volume of the cylinder

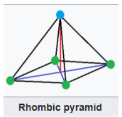
$V_{cyl} = \frac{1}{3} \cdot \sqrt{\frac{1}{6}} \cdot S \cdot \sqrt{\frac{S}{\pi}}$.

With the volume of the sphere, and $r = \frac{1}{\sqrt{6}} \cdot \sqrt{\frac{S}{\pi}} \Longrightarrow L = \sqrt{\frac{2}{3}} \cdot \sqrt{\frac{S}{\pi}}$, for the cylinder:

$\frac{V_{sphere}}{V_{cyl}} = \sqrt{\frac{3}{2}} > 1$, the Isoepifaeic Inequality.

§ 4.4 The Isoepifaeic Inequality of a Pyramid with rhombic base for a given Surface

In the Figure 4.3 below the rhombic pyramid is pictured(see insert)



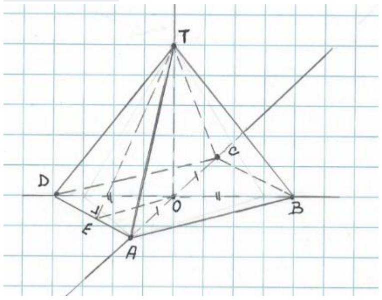


Figure 4.3 The Rhombic Pyramid

Top view:

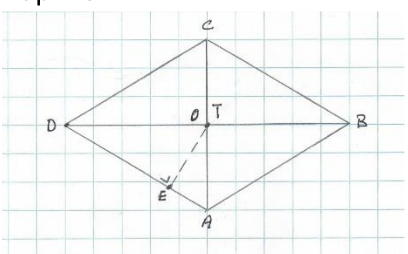


Figure 4.4 Top view of Figure 4.3

In Figure 4.4 the top view of the pyramid is presented.

$ABCD$ is the rhombic base.

Denote $DB \to w, AC \to h, OE \to x, EA \to y, ET \to z$ and $OT \to t$

In the above top view TE/OE represent the perpendicular of the ΔTAD and ΔOAD respectively.

$x^2 = \frac{1}{4} \cdot h^2 - y^2,$

and

$x^2 = \frac{1}{4} \cdot w^2 - (\frac{1}{2}\sqrt{h^2 + w^2} - y)^2.$

Then,

$y = \frac{1}{2} \frac{h^2}{\sqrt{h^2+w^2}},$

and

$x = \frac{1}{2} \frac{hw}{\sqrt{h^2+w^2}}.$

Furthermore $z = \sqrt{x^2 + t^2}$.

Area $\Delta TAD \to \frac{1}{2} z \cdot |DA| = \frac{1}{2}\sqrt{x^2 + t^2} \cdot \frac{1}{2}\sqrt{h^2 + w^2} = \frac{1}{4}\sqrt{t^2(h^2 + w^2) + \frac{1}{4} h^2 w^2}.$

The area of the pyramid is

$S = \frac{1}{2} \cdot hw + \sqrt{t^2(h^2 + w^2) + \frac{1}{4} h^2 w^2}$.

The volume of the pyramid is

$V = \frac{1}{3} \cdot \frac{1}{2} \cdot \ hw \cdot t.$

To investigate the isoepifaeic inequality, we express t into S and plug the result into V:

$t = [\frac{S^2 - hwS}{h^2 + w^2}]^{1/2},$

$V = \frac{1}{6}\ hw[\frac{S^2 - hwS}{h^2 + w^2}]^{1/2}.$

Next, we make the preceding expression dimensionless with S:

$V' = \frac{V}{S^{3/2}}$, $h' = \frac{h}{S^{1/2}}$, and $w' = \frac{w}{S^{1/2}}$.

Plug this result in the expression for V and after dropping the primes we obtain:

$V = \frac{1}{6}\ hw[\frac{1 - hw}{h^2 + w^2}]^{1/2}$

To obtain a stationary point, we determine the results of $\frac{dV}{dh}$ and $\frac{dV}{dw}$.

Note: in 3-D, $V(h, w)$ is a symmetrical function with the plane of symmetry $h = w$, and the V-axis.

$\frac{dV}{dh} = 0 \Longrightarrow 2w - 3hw^2 - h^3 = 0$,

and

$\frac{dV}{dw} = 0 \Longrightarrow 2h - 3h^2w - w^3 = 0$.

Multiply $\frac{dV}{dh} = 0$, with h and $\frac{dV}{dw} = 0$, with w.

Equate both resulting equations:

$h^4 = w^4 \Longrightarrow h = w \rightarrow$ the rhombic base changes into a square base.

Then, with $h = w$

$V = \frac{1}{6}\ hw[\frac{1 - hw}{h^2 + w^2}]^{1/2} \Longrightarrow V = \frac{\sqrt{2}}{12} h(1 - h^2)^{1/2}$.

In Figure 4.5 a plot of V is presented.

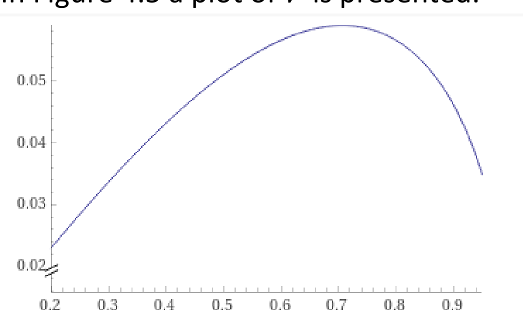


Figure 4.5 V(h) as a function of h

A maximum indeed.

Now, wit $V = \frac{\sqrt{2}}{12} h(1 - h^2)^{1/2}$

$\frac{dV}{dh} = 0 \Longrightarrow h = \frac{1}{2}\sqrt{2} = w.$

The sides of the square base are, dimensionless, Figure 4.4

$DA = AB = BC = CD = \frac{1}{2}.$

The height of the pyramid, dimensionless, with $h = \frac{1}{2}\sqrt{2}$

$$t = \frac{\sqrt{2}}{2}\frac{(1-h^2)^{1/2}}{h} = \frac{1}{2}\sqrt{2}\,.$$

The result is a pyramid with a square base with sides $\frac{1}{2}$ and height $\frac{1}{2}\sqrt{2}$.

The sides of the pyramid are four congruent isosceles triangles.

The maximum volume V with the aforementioned numbers:

$$V_{max} = \frac{\sqrt{2}}{24}\,.$$

The dimensionless volume of the sphere with equal surface S is

$$V_{sphere} = \frac{1}{6\sqrt{\pi}}\,.$$

Then

$$\frac{V_{sphere}}{V_{max}} = \frac{1}{6\sqrt{\pi}}\cdot\frac{24}{\sqrt{2}} = \frac{4}{\sqrt{2\pi}} \cong 1.6.$$

Note.

One could wonder whether a pyramid with square base and four congruent equilateral triangles is not the structure with the maximum dimensions, i.e. surface.

Let us find out.

The surface of this pyramid, with a 1×1 basis is

$$S = 1 + \sqrt{3},$$

and

$$V = \frac{1}{6}\sqrt{2}.$$

Now the sphere with the same surface

$$4\pi r^2 = S = 1 + \sqrt{3} \rightarrow r = \sqrt{\frac{1+\sqrt{3}}{4\pi}}\,.$$

$$V_{sphere} = \frac{4}{3}\pi r^3 = \frac{1}{6}\frac{(1+\sqrt{3})^{3/2}}{\sqrt{\pi}}\,.$$

$$\frac{V_{sphere}}{V} = \frac{(1+\sqrt{3})^{3/2}}{\sqrt{2\pi}} = \sqrt{\frac{5+3\sqrt{3}}{\pi}} \cong 1.8\,.$$

§ 5 The Isoepifaeic (In)equality

Can we proof the conjecture: The inequality: for a given surface S, the sphere to have the biggest capacity or volume V.

So, the inequality reads

$$V \leq \frac{1}{6\sqrt{\pi}}S^{3/2}.$$

In order to proof the conjecture, we first examine a couple of examples.

§ 5.1 Two half Spheres with different radii

This example is illustrated in the Figure below.

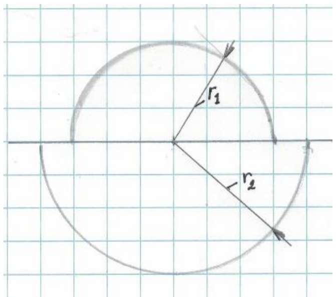


Figure 5.1 Two half Spheres

The surface of the two half spheres and the area of the ring with width $|r_2 - r_1|$

$S = 2\pi r_1^2 + 2\pi r_2^2 + \pi|r_2^2 - r_1^2|$.

The volume

$V = \frac{2}{3}\pi r_1^3 + \frac{2}{3}\pi r_2^3$.

Obviously, it is sufficient to investigate $r_2 \geq r_1 \Longrightarrow$

$\Longrightarrow S = \pi r_1^2 + 3\pi r_2^2$

We investigate the maximum value of V for a given surface S.

Express r_1 in terms of S and r_2:

$r_1 = \sqrt{\frac{S}{\pi} - 3r_2^2}$.

Then,

$V = \frac{2}{3}\pi(\frac{S}{\pi} - 3r_2^2)^{3/2} + \frac{2}{3}\pi r_2^3$.

We can make this expression dimensionless with S and proceed with the analysis.

There is no need to do this.

With

$S = 2\pi r_1^2 + 2\pi r_2^2 + \pi|r_2^2 - r_1^2|$,

the maximum value for r_1 and r_2 respectively is obtained for an assumed given value of S when $|r_2^2 - r_1^2| = 0 \to r_1' = r_2' \to$ a new r.

A sphere with radius r:

$4\pi r^2 = S = \pi r_1^2 + 3\pi r_2^2$.

Hence,

$r = \frac{1}{2}\sqrt{r_1^2 + 3r_2^2}$.

Note:

with $4\pi r^2 = S = \pi r_1^2 + 3\pi r_2^2$,

it becomes clear, by decreasing r_2 and increasing r_1, a circle with radius r is obtained.

This compares with the 2-D case changing a concave perimeter into a convex perimeter with the same perimeter.

§ 5.2 Four quarter Spheres with different radii

This example is illustrated in the Figure below.

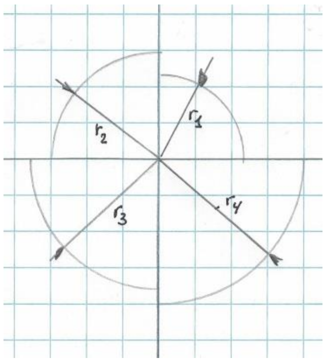


Figure 5.2 Four quarter Spheres

The surface of these four segments is:

$S = \pi \sum_{n=1}^{4} r_n^2 + \frac{1}{2}\pi \sum_{n=1}^{3} |r_{n+1}^2 - r_n^2| + \frac{1}{2}\pi |r_1^2 - r_4^2|.$

The maximum value for $\pi \sum_{n=1}^{4} r_n^2$ is obtained with a given value of S when $\left|r_{i+1}^2 - r_i^2\right| = 0, i = 1,2,3$ and $|r_1^2 - r_4^2| = 0$.

A sphere with radius r:

$4\pi r^2 = S = \pi \sum_{n=1}^{4} r_n^2 + \frac{1}{2}\pi \sum_{n=1}^{3} |r_{n+1}^2 - r_n^2| + \frac{1}{2}\pi |r_1^2 - r_4^2|,$

$r = \frac{1}{2}\sqrt{\sum_{n=1}^{4} r_n^2 + \frac{1}{2}\sum_{n=1}^{3} |r_{n+1}^2 - r_n^2| + \frac{1}{2}|r_1^2 - r_4^2|}.$

We can always position the quarter spheres as presented in Figure 5.2.

Then, we obtain for r:

$r = \frac{1}{2}\sqrt{2r_4^2 + r_3^2 + r_2^2}$

§ 5.3 A 3-D Body with a smooth, closed Surface

This 3-D body, with surface area S, can be divided in n segments, like the four segments in Figure 5.2.

$S = \frac{4\pi}{n} \sum_{k=1}^{n} r_k^2 + \frac{1}{2}\pi \sum_{1=1}^{n-1} |r_{i+1}^2 - r_i^2| + \frac{1}{2}\pi |r_1^2 - r_n^2|.$

The maximum value of the volume of this body is again a sphere, with radius

$r = \frac{1}{2}\sqrt{\frac{4}{n}\sum_{k=1}^{n} r_k^2 + \frac{1}{2}\sum_{1=1}^{n-1} |r_{i+1}^2 - r_i^2| + \frac{1}{2}|r_1^2 - r_n^2|}.$

However, this 3-D object is not a general object, and the approximation with n segments is too rough an approximation.

Hence, a bit more has to be done.

This can be done by representing the 3-D object with rhombic pyramids of which the base describes the surface of this smooth 3-D object. The height of the pyramid is equal to the distance of the origin O of the Cartesian coordinate system: r.[3]

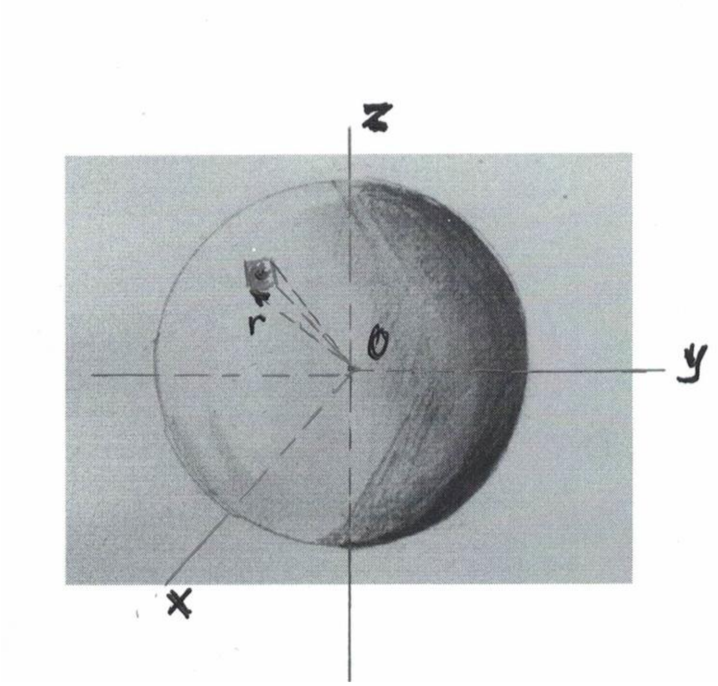


Figure 5.3 3-D object represented by a large number of rhombic pyramids.

The base area of the pyramid is A_{ij} , the height is r_{ij}.
The pyramid is surrounded by four pyramids: $A_{i-1,j}, A_{i-1,j-1}, A_{i+1,j}$ and $A_{i+1,j+1}$. All pyramids with the same solid top angle $\Delta\theta$. The area $A_{ij} = r_{ij}^2 \cdot \Delta\theta$.
The differences in the height between r_{ij} and the heights of the surrounding pyramids are denoted by $\Delta A_{i-1,j}, \Delta A_{i-1,j-1}, \Delta A_{i+1,j}$, and $\Delta A_{i+1,j+1}$.
The surface S of this 3-D object is approximated by
$S = \sum_{i,j}^{n} A_{ij} + \sum_{i,j=1}^{n}(\Delta A_{i-1,j} + \Delta A_{i-1,j-1} + \Delta A_{i+1,j} + \Delta A_{i+1,j+1})$.
Or,
$S = \sum_{i,j}^{n} r_{ij}^2 \cdot \Delta\theta + \sum_{i,j=1}^{n}(\Delta A_{i-1,j} + \Delta A_{i-1,j-1} + \Delta A_{i+1,j} + \Delta A_{i+1,j+1})$.
Since $\sum_{i,j=1}^{n}(\Delta A_{i-1,j} + \Delta A_{i-1,j-1} + \Delta A_{i+1,j} + \Delta A_{i+1,j+1}) > 0$,
r_{ij} becomes its maximum value for
$\sum_{i,j=1}^{n}(\Delta A_{i-1,j} + \Delta A_{i-1,j-1} + \Delta A_{i+1,j} + \Delta A_{i+1,j+1}) = 0$.
Then $r_{ij} = r$, the same for all pyramids.
For $n \to \infty$, The volume of the 3-D object becomes:
$V = \int_0^{4\pi} \int_0^{r} (r')^2 \, dr' d\theta = \frac{4}{3}\pi r^3$, a sphere.

[3] Figure 5.3 reminded me of the Manhattan Project: "*Von Neumann suggested a better design for the implosion device, consisting of wedge-shaped charges arranged around the plutonium*"., Bhattacharya (2021).

§ 6 Conclusions

In the proof of the isoperimetric inequality, use has been made of a simple perturbation of a circle with radius R_1. The perturbation is $\Delta R_1 \sin\theta$. The curve can be more perturbed by using Fourier series expansion. To keep the closed curve convex becomes more complicated. However, the principle is demonstrated with the first term, $\Delta R_1 \sin\theta$, of the Fourier expansion. The isoperimetric inequality is demonstrated with the examples of the equilateral triangle up till the regular polygon. The equality has been proven in two ways:

- by a perturbed circle where the first term of a Fourier expansion has been used.
- a closed curve approximated by a set of different isosceles triangles, grouped around the origin of a cartesian frame, with equal top angles.

Furthermore, the three-dimensional case , denoted isoepifaeic inequality is investigated.
It appears , the sphere to be the largest volume for a given surface of a smooth 3-D object.

§ 5 Literature

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