de Fermat's Last Theorem in the marginal line of Diophantus.

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Content

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Abstract

Another proof of de Fermat's last theorem is presented.
The results for $n = 3{,}4$ are used. A set of values of n , integer multiples of $\{3{,}4\}$, is obtained for which no integer solutions can be found.
A concise proof of de Fermat's last theorem, is obtained by using Pythagorean Triples.
Use has been made of polynomials and proofs by contradiction.
n is prime, in $x^n + y^n = z^n$, is key to the proof of Fermat's theorem.
In the limit $n = 2$, the proof of the existence of Pythagorean triples is given.

§1. Introduction.

'Fermat's equation: $x^n + y^n = z^n$ *,* **(1.1)**

has no solutions for $n \geq 3'$.

A statement by Andrew Wiles written on the black board after the presentation of the

conclusions concerning equation **(1.1).**
Is there another proof of the last theorem of de Fermat, fitting into the margin of de Fermat's copy of Diophantus?
de Fermat mentioned his proof not to fit in the width of the margin of his Diophantus copy. An Ockham's razor, sort of. This stimulated me to investigate de Fermat's last theorem.
It all started with $n = 2$ and

$$a^2 + b^2 = c^2. \qquad \textbf{(1.2)}$$

For equation **(1.2)** there are an infinite number of solutions for a, b and c as positive integers; $\{a, b \text{ and } c \in \mathbb{P}\}$. The Pythagorean Triples.
This equation, **(1.2)**, is discussed in some detail in Noordzij.
For $n = 3$ and $n = 4$, the proof of the non-existence of integer solutions has been given with the so-called method of '*Descente infinie*' (Giorello, G., et al).

§2. Notation

I assume a solution of **(1.1)** in terms of integers x, y and z : a so-called Fermat Triple.
I denominate the solution a "Fermat" triple: $x = A, y = B$ and $z = C$.
$[A, B, C]$ is assumed to be the triple
A, B and C being positive integers. A is an odd and B is assumed to be even and C must be odd.
So, $\{\exists A, C \in \mathbb{P}|$ and odd$\}$ and $\{\exists B \in \mathbb{P}|$ and even$\}$; $\{\forall n \in \mathbb{P}|n \geq 3\}$.
A, B and C are relative- or co-prime.
Factorise A , *the fundamental theorem of arithmetic*:

$$A = 3^{n_1}5^{n_2}7^{n_3}.....p_i^{n_i}....= \prod_{i=1}^{\infty} p_i^{n_i}; \qquad \textbf{(2.1)}$$

$\{n_i \in \mathbb{N} \cup \emptyset\}$ and $\{p_i \in \mathbb{P}\}$,where $\mathbb{P}$ is the set of positive,
I choose a concise notation and write for A:

$$A = P_j P_l (\equiv \prod_{j=1}^{\infty} p_j^{n_j} \prod_{l=1}^{\infty} p_l^{n_l}), \qquad \textbf{(2.2)}$$

and $l \neq j$; P_j and P_l are co-prime.

$$n \geq 3: n = 2^{m_0}3^{m_1}5^{m_2} \ldots .. p_k^{n_k} \ldots . = \prod_{k=0}^{\infty} p_k^{m_k}, \qquad \textbf{(2.3)}$$

$\{m_k \in \mathbb{P}\}, \{n \in \mathbb{P}\}$, $\{p_k \in \mathbb{P}\}$ and $\{m \in \mathbb{P}\}$.
Furthermore I will use(see e.g. Spivak):
$\forall$, the al kwantor meaning : "for all". So, $\{\forall n \in \mathbb{P}\}$ →for all n belonging to the set of natural numbers larger than zero.
$\exists$, the existential kwantor: "there exists". So, $\{\exists n \in \mathbb{P}\}$ →there exists a n belonging to the set of natural numbers larger than zero.

§3. de Fermat's last Theorem.

3.1 $n = 3$, $n = 4$

First $n = 3$.
As mentioned in the Introduction, there exists no Fermat triple for $n = 3$. I shall make use of this result to proof no Fermat triples exist for integer multiples of $n = 3 \rightarrow n = m \cdot 3$.

So,
$\{\forall\ m \in \mathbb{P}|n = m \cdot 3\ \}$ there exists no Fermat Triple.

Proof
$\{\exists[A, B, C] \in \mathbb{P}\}$.
We have, with $n = m \cdot 3$,
$A^n + B^n = C^n \Leftrightarrow A^{m3} + B^{m3} = C^{m3} \Leftrightarrow (A^m)^3 + (B^m)^3 = (C^m)^3$,
where $\{A \in \mathbb{P}\}$.
With the result of the '*Descente infinie*' (Giorello, G., et al) for $n = 3 \rightarrow$
$\rightarrow \{\nexists(B^m, C^m) \in \mathbb{P}\}$.
So, a for a given integer $A \rightarrow \{B^m, C^m \notin \mathbb{P}\}$.
Consequently, $\{B, C \notin \mathbb{P}\}$.
This contradicts $\{\exists(B, C) \in \mathbb{P}\}$.
Consequently, $\{\nexists[A, B, C] \in \mathbb{P}|\ n = 3 \rightarrow n = m \cdot 3\}$.
End of proof.

Next, we use $n = 4$ and the non-existence of Fermat Triples.
As mentioned in the Introduction, there exists no Fermat triple for $n = 4$. I shall make use of this result to proof no Fermat triples exist for integer multiples of $n = 4 \rightarrow n = m \cdot 4$.
$A^n + B^n = C^n \Leftrightarrow A^{m4} + B^{m4} = C^{m4} \Leftrightarrow (A^m)^4 + (B^m)^4 = (C^m)^4$.
For a given integer $A \rightarrow \{(B^m, C^m) \notin \mathbb{P}\}$.
Completely like the above proof for $n = 3$:
$\{(B, C) \notin \mathbb{P}\} \rightarrow \{\nexists[A, B, C] \in \mathbb{P}|\ n = 4 \rightarrow n = m \cdot 4\}$.
Hence for the set of integer multiples of $n = \{3,4\}$, there are no Fermat triples.
Now, let's look again at
$A^{m3} + B^{m3} = C^{m3}$,
and $\{\forall m \in \mathbb{P}\}$.
For a given integer $A \rightarrow \{\nexists[A, B, C] \in \mathbb{P}|n = 3 \rightarrow n = m \cdot 3\}$.

Proof
$A^n + B^n = C^n \Leftrightarrow A^{m3} + B^{m3} = C^{m3}$.
For a given integer $A \rightarrow \{B, C \notin \mathbb{P}\}$. See the above proof for $n = m \cdot 3$.
Now,
$A^n + B^n = C^n \Leftrightarrow A^{m3} + B^{m3} = C^{m3} \Leftrightarrow (A^m)^3 + (B^m)^3 = (C^m)^3 \Leftrightarrow (A^3)^m + (B^3)^m = (C^3)^m$.
Again: $\{\forall A \in \mathbb{P}\} \rightarrow \{\nexists B, C \in \mathbb{P}\}$.
Then,
$\{B^3, C^3 \notin \mathbb{P}\} \rightarrow \{\forall m \in \mathbb{P}\} \rightarrow \{\nexists[A, B, C] \in \mathbb{P}\}$ for $\{\forall n \in \mathbb{P}|n \geq 3, n = m \cdot 3\}$.
End of Proof

No new information is created?
Well, we have
$(A^3)^m + (B^3)^m = (C^3)^m$.
We have proven for a given integer $A \Longrightarrow \{B^3, C^3 \notin \mathbb{P}\}$.
So, with $m = 2$, a Pythagorean Triple cannot constitute a set of integers $\{\exists[A, B, C] \in \mathbb{P}\}$ written as $[a^3, b^3, c^3]$ and $\{a, b, c \in \mathbb{P}\}$.
Furthermore,
$A^3 \equiv D, B^3 \equiv E$ and $C^3 \equiv F$,
$(D)^m + (E)^m = (F)^m$.
Then,

$\{E, F \notin \mathbb{P}\}$, for $\{\forall m \in \mathbb{P} | m \geq 3\}$.
So, no Fermat Triples exist.

3.2 Factorization of C^n-B^n=A^n

$[A, B, C]$ is assumed to be a triple, $\{\exists [A, B, C] \in \mathbb{P}\}$, as defined in §2, a Fermat triple.
In the subsequent sections, I set $\{A \in \mathbb{P}|$ odd$\}$ and $\{C - B \in \mathbb{P}|$ odd$\}$. We needed to proof $\{\exists C, B \in \mathbb{P}\}$.

(1.1) :

$A^n + B^n = C^n$ or

$$A^n + B^n = C^n \rightarrow C^n - B^n = A^n. \quad \textbf{(3.1)}$$

Then **(3.1)** gives:

$$(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = A^n . \quad \textbf{(3.2)}$$

Proof :
$(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = C^n - B^n$?
$(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = \sum_{k=0}^{n-1} C^{n-k} B^k - \sum_{k=0}^{n-1} C^{n-1-k} B^{k+1} \rightarrow$
$\rightarrow C^n + \sum_{k=1}^{n-1} C^{n-k} B^k - B^n - \sum_{k=0}^{n-2} C^{n-1-k} B^{k+1}$.
Now change the summation index of $\sum_{k=1}^{n-1} C^{n-k} B^k$,
$k - 1 = r$ and :
$(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = C^n - B^n + \sum_{r=0}^{n-2} C^{n-1-r} B^{r+1} - \sum_{k=0}^{n-2} C^{n-1-k} B^{k+1}$.
Hence $\sum_{r=0}^{n-2} C^{n-1-r} B^{r+1} - \sum_{k=0}^{n-2} C^{n-1-k} B^{k+1} = 0$.
Consequently $(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = C^n - B^n$.
End of Proof.

With the presumption of A, B and C being a Fermat triple, $(C - B)$ and the summation between brackets in **(3.2)** are positive integers. Both $(C - B)$ and the summation in **(3.2)** are integer factors of A^n, consisting of products of powers of prime numbers, **(2.2)**, since A is a product of prime numbers. *The fundamental theorem of arithmetic*.
$(C - B)$ is an integer factor of $A^n = P_j^n P_l^n$:

Proof
Let $(C - B)$ be no integer factor of A^n. Consequently $\frac{A^n}{(C-B)}$ is not an integer.
Then, **(3.2)**:
$\sum_{k=0}^{n-1} C^{n-1-k} B^k = \frac{A^n}{(C-B)}$,
is no integer.
Since I assumed a Fermat Triple to exist:
$\sum_{k=0}^{n-1} C^{n-1-k} B^k$,
must be an integer.
So, for a Fermat triple to exist, $(C - B)$ must be an integer factor of A^n.
Hence,
$\sum_{k=0}^{n-1} C^{n-1-k} B^k$, the summation in **(3.2)**, is an integer factor of A^n.
Then, the assumption $(C - B)$ be no integer factor of A^n is contradicted.
Note: remember *"the fundamental theorem of arithmetic"*:
$(C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k)$ has to be a product of (powers of) prime numbers.
End of proof.

I set: $C - B = P_j^n$, see **(2.2)**, **(3.3)**
to be an integer factor of A^n is a condition to find Fermat triples.

The summation in **(3.2)** constitutes the other factor of A^n, $\{\exists P_l^n \in \mathbb{P} | P_l^n > 0 \text{ and odd}\}$. Furthermore $P_j^n < A$.

Proof:
$C - B = P_j^n (\equiv A_1, A_1 > 0)$,
and $C^n \equiv (B + A_1)^n = \sum_{k=0}^{n} \binom{n}{k} {A_1}^k B^{n-k} = B^n + A_1^n + \sum_{k=1}^{n-1} \binom{n}{k} {A_1}^k B^{n-k} = B^n + A^n \rightarrow$
$\rightarrow A_1^n + \sum_{k=1}^{n-1} \binom{n}{k} {A_1}^k B^{n-k} = A^n$.
Then, with the positive terms of the polynomial, it's about the assumption of the existence of a Fermat Triple:
$A_1^n = A^n - \sum_{k=1}^{n-1} \binom{n}{k} {A_1}^k B^{n-k} \rightarrow A_1 = P_j^n (= C - B) < A$.
With **(2.2)**: $P_j^n < P_j P_l \rightarrow P_j^{n-1} < P_l$.
End of Proof.

$(C - B)$ and $(\sum_{k=0}^{n-1} C^{n-1-k} B^k)$ do not have an integer factor in common. Else,
$C^n - B^n = A^n = (C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k)$, can be divided by the n^{th} power of this common integer factor.
Assuming a Fermat Triple to exist.
Suppose C and B have a factor $p, \{p \in \mathbb{P}\}$, in common:
$A^n = p(C - B)(\sum_{k=0}^{n-1} p^{n-1-k} C^{n-1-k} p^k B^k) = p^n (C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k)$.
Then, with the assumption a Fermat Triple exists, A^n contains the factor p^n. Consequently, $A^n + B^n = C^n$ can be divided by p^n.
Note: in,
$A^n = p(C - B)(\sum_{k=0}^{n-1} p^{n-1-k} C^{n-1-k} p^k B^k) = p^n (C - B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k)$
C and B are new numbers C and B.

Substitution of $A_1 (\equiv C - B)$ into **(3.1)** gives a polynomial in B.
So, given A, and $A_1 (0 < A_1 < A)$:
$(B + A_1)^n - B^n = A^n$,
or
$\sum_{k=0}^{n} \binom{n}{k} {A_1}^k B^{n-k} = A^n + B^n$.
The polynomial in B is:

$$\sum_{k=1}^{n-1} \binom{n}{k} {A_1}^k B^{n-k} - (A^n - A_1^n) = 0, \qquad \textbf{(3.4)}$$

or

$$B^{n-1} + \sum_{k=2}^{n-1} \frac{1}{n} \binom{n}{k} A_1^{k-1} B^{n-k} - \frac{1}{n A_1}(A^n - A_1^n) = 0 \ . \qquad \textbf{(3.5)}$$

with **(2.2)**

$$B^{n-1} + \sum_{k=2}^{n-1} \frac{1}{n} \binom{n}{k} P_j^{n(k-1)} B^{n-k} - \frac{1}{n}\left[P_l^n - P_j^{n(n-1)}\right] = 0, \qquad \textbf{(3.6)}$$

The expression between square brackets is positive, since $A^n > A_1^n$.
P_l and P_j do not have integer factors in common (co-prime), n can be an integer factor of P_l or of P_j.
Furthermore, n can be an integer factor of $\left[P_l^n - P_j^{n(n-1)}\right]$. I will pay attention to this case in the next section.

3.3 Fermat Triples and n is Prime

$\{\exists[A, B, C] \in \mathbb{P}\}$.

To confirm this assumption, the question to be answered is: can I obtain integer roots for B from **(3.6)** for $\{\forall(C-B) \in \mathbb{P}\}$?

Set n to be a prime number, and n can be a factor of P_l or P_j. Not of both.

Then, could B be an integer root of the polynomial, with **(3.6)**:

$$B^{n-1} + \sum_{k=2}^{n-1} \frac{1}{n}\binom{n}{k} P_j^{n(k-1)} B^{n-k} = \frac{1}{n}\left[P_l^n - P_j^{n(n-1)}\right] ? \quad \textbf{(3.7)}$$

For n is prime, the factor $\{\frac{1}{n}\binom{n}{k} = \frac{(n-1)!}{k!(n-k)!} \in \mathbb{P} \mid 1 \leq k \leq n-1\}$.

Proof:

$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k!(n-k)!}$ is an integer, Newtons binomial distribution.

Since n is prime, $k!$ and $(n-k)!$ are no integer factors of n.

$k!$ and $(n-k)!$ have to be integer factors of $(n-1)!$.

Hence,

$\{\frac{1}{n}\binom{n}{k} = \frac{(n-1)!}{k!(n-k)!} \in \mathbb{P} | 1 \leq k \leq n-1\}$.

End of Proof.

Does $[A, B, C]$ exist for n is prime and $n \geq 3$?

Proof

(3.7) :

$B^{n-1} + \sum_{k=2}^{n-1} \frac{(n-1)!}{k!(n-k)!} P_j^{n(k-1)} B^{n-k} = \frac{1}{n}\left[P_l^n - P_j^{n(n-1)}\right]$.

So,

for n is prime, all the coefficients, $\frac{(n-1)!}{k!(n-k)!} P_i^{n(k-1)}$ of B^{n-k} in the left hand side of **(3.7)** are integers. Since B is assumed to be an integer, the left-hand side is an integer. The right-hand side is not an integer for:

- n to be a factor of P_l or P_j. Not of both. Since, $(C-B)$ and $(\sum_{k=0}^{n-1} C^{n-1-k} B^k)$ do not have an integer factor in common.

- n not to be an integer factor of $\left[P_l^n - P_j^{n(n-1)}\right]$.

Hence,

B is not an integer.

So, there exists no Fermat Triple for n is prime and $n \geq 3$

End of Proof

Is it necessary to analyse the case where n is an integer factor of $\left[P_l^n - P_j^{n(n-1)}\right]$?

Well, no. I can always **choose n to be an integer factor of A** for $\{\forall n \in \mathbb{P} | n \geq 3\}$.

Then again, $\frac{1}{n}\left[P_l^n - P_j^{n(n-1)}\right]$ is not an integer. So, n can be a factor of P_l or P_j . Not of both.

In the following analysis I use $\{\frac{1}{n}\left[P_l^n - P_j^{n(n-1)}\right] \notin \mathbb{P}\}$.

Now, I assume $\{B \in \mathbb{Q} | B > 0\}$. B can be expressed as a quotient of (powers of) prime numbers, P_s and P_h respectively, with no common factors and $P_s > P_h$:

$B = \frac{P_s}{P_h}$.

Plug this expression for B_1 into **(3.1)**:

$$A^n + \left(\frac{P_s}{P_h}\right)^n = C^n, \quad \textbf{(3.8)}$$

for $\{\exists(C-B) \in \mathbb{P}\}$ with $\{A \in \mathbb{P} | A, (\mathbf{2.2})\}$.

Consequently, $\{C \in \mathbb{Q}\}$
I conclude, for any positive integer value of $(C - B)$ with $\{A \in \mathbb{N} > 0, (\mathbf{2.2})\} \rightarrow$
$\rightarrow \{B, C \in \mathbb{Q}\}$.
Now I will proof B to be Algebraic irrational.

Proof
(3.8):
Multiply $A^n + (\frac{P_s}{P_h})^n = C^n$, with $P_h^n \rightarrow (AP_h)^n + P_s{}^n = (CP_h)^n, P_s > B_1$.
We can write:
$(AP_h)^n + P_s{}^n = (CP_h)^n \rightarrow (A_r)^n + (B_r)^n = (C_r)^n$.
Use the analysis of **(3.7)** and obtain $B_r(= P_s > B)$ not to be an integer. So, B cannot be a rational number.
We already proved $\{B \notin \mathbb{P}\}$. Consequently, B is Algebraic irrational.
Note: we could have continued with B_r again to be a rational number and be presented as a quotient of prime numbers. This process can be continued ad infinitum. Then an "**Ascent Infinite**" is created: $A_r \rightarrow \infty$ and $C_r \rightarrow \infty$.
This proof seems to me trivial. Since, B is assumed rational, so is C and consequently A.
However, this contradicts the assumptions.
End of Proof.

There are no Fermat Triples for n is prime.
n is prime is key to the solutions proven in the following section.

3.4 Fermat Triples n a product of powers of odd and even prime numbers

$\{\exists[A, B, C] \in \mathbb{P}\}$.
To confirm this assumption, the question to be answered is: can I obtain integer roots for B from **(3.6)** for any positive integer value of $(C - B)$?
As mentioned before, n is prime is key to $\{\nexists[A, B, C] \in \mathbb{P}\}$.
$A^n + B^n = C^n$, **(3.1)** .
For n to be a product of prime numbers, there exists no Fermat Triple.

Proof
n can be expressed as a product of powers of odd prime numbers denoted by P_k.
Plug this into **(3.1)**:
$A^{P_k} + B^{P_k} = C^{P_k}$. **(3.9)**
I can always choose a prime number p being a factor of P_k.
So,
$P_k = pP_m$. **(3.10)**
Substitute $P_k = pP_m$ into **(3.9):**
$(A^{P_m})^p + (B^{P_m})^p = (C^{P_m})^p$. **(3.11)**
In section 3.2, I proved B^{P_m} not to be an integer for p to be a prime number > 2.
Hence, for (A^{P_m}) assumed to be an integer, (C^{P_m}) and (B^{P_m}) are no integers.
In section 3.2, I proved B^{P_m} to be Algebraic irrational.
Assume B to be an integer→$\{B^{P_m} \in \mathbb{P}| > 0\}$ →a contradiction.
Hence, B is Algebraic irrational.

End of proof.

No Fermat Triples for n **is a product of powers of odd and even prime numbers.**

3.5 Fermat Triples n is even and a product of powers of prime numbers

$\{\exists[A,B,C] \in \mathbb{P}\}$.
In section 3.4, I dealt with these values of n without powers of the prime number 2. This makes no difference.
Here , I present a slightly different approach.
To confirm the assumption of Fermat Triples to exist, the question to be answered is: can I obtain integer roots for B from **(3.6)** for any positive integer value of $(C-B)$?
- n is even and $m \neq 0$, see **(2.3).**
The equation to start with is **(3.2)**:
$(C-B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = A^n$.
Note: assuming a Triple to exist, applying *the fundamental theorem of arithmetic*, $(C-B)$ and $(\sum_{k=0}^{n-1} C^{n-1-k} B^k)$ are comprising the factorization.

First, $n = 2$.
Proof of the existence of Pythagorean triples.
Then,
$(C-B)(C+B) = A^2$.
With $(C-B) = A_1$, section 3.1:
$(C-B)(C+B) = A^2 \rightarrow A_1(2B + A_1) = A^2 \rightarrow B = \frac{1}{2}\left(\frac{A^2}{A_1} - A_1\right) = \frac{1}{2}\left(\frac{A^2 - A_1^2}{A_1}\right)$.
Using A_1 to be an integer factor of A^2, see also proof in section 3.1:
$\{(\frac{A^2}{A_1} - A_1) \in \mathbb{P} | \left(\frac{A^2}{A_1} - A_1\right)$, even $\}$ and $A_1 < A \rightarrow B$ is an integer.
Furthermore,
$C - B = A_1 \rightarrow C = B + A_1 = \frac{1}{2}\left(\frac{A^2}{A_1} + A_1\right) = \frac{1}{2}\left(\frac{A^2 + A_1^2}{A_1}\right)$.
$\{\frac{1}{2}\left(\frac{A^2}{A_1} + A_1\right) \in \mathbb{P} | \left(\frac{A^2}{A_1} - A_1\right)$, odd$\}$ $\rightarrow C$ is an integer.
Consequently, Pythagorean triples exist.
In addition,
$A^2 - A_1^2$ comprises a factor 4 and $A^2 + A_1^2$ a factor 2.
This leads to the conclusion: B must be even and C must be odd.
End of Proof.

Next,
$n = 2(s+1)$,$\{s \in \mathbb{P}\}$.
Then **(3.2)** can be factorized:
$(C-B)(C+B)[\sum_{k=0}^{s}(C^2)^{s-k} (B^2)^k] = A^n$. **(3.13)**

Proof:
$(C+B)[\sum_{k=0}^{r}(C^2)^{s-k} (B^2)^k] = \sum_{k=0}^{n-1} C^{n-1-k} B^k$.

$(C+B)[\sum_{k=0}^{s}(C^2)^{s-k}\,(B^2)^k] = \sum_{k=0}^{r} C^{2s-2k+1}\,B^{2k} + \sum_{k=0}^{r} C^{2s-2k}\,B^{2k+1}$. **(3.14)**

Set $2k = k'$.

Then

$k' = 0,2,\ldots\ldots,2r$.

Plug k' into **(3.14),** with $2r = n-2$:

$\sum_{k'=0}^{n-2} C^{n-1-k'}\,B^{k'} + \sum_{k'=0}^{n-2} C^{n-2-k'}\,B^{k'+1}$. **(3.15)**

Set $k'+1 = t$ in the second summation of **(3.15)** $\rightarrow t = 1,3,\ldots\ldots..,n-1$:

$\sum_{k'=0}^{n-2} C^{n-1-k'}\,B^{k'} + \sum_{t=1}^{n-1} C^{n-1-t}\,B^{t}$. **(3.16)**

Compare the range of the dummy variables k' and t in **(3.16),** and the two sums in **(3.16)** can be combined:

$\sum_{k'=0}^{n-2} C^{n-1-k'}\,B^{k'} + \sum_{t=1}^{n-1} C^{n-1-t}\,B^{t} = \sum_{k=0}^{n-1} C^{n-1-k}\,B^{k}$.

End of proof.

With $\{\exists[A,B,C] \in \mathbb{P}\}$, $(C+B)$ and the summation between brackets in **(3.13)** are positive integers.

$\{(C+B) \in \mathbb{P}\}$ in **(3.13)**, is an integer factor of $(\sum_{k=0}^{n-1} C^{n-1-k}\,B^{k})$, **(3.2)**.

Proof

Let $(C+B)$ not be an integer factor of $\sum_{k=0}^{n-1} C^{n-1-k}\,B^{k} = \frac{A^n}{C-B}$.

Consequently, $\sum_{k=0}^{s}(C^2)^{s-k}\,(B^2)^k = \frac{A^n}{(C-B)(C+B)}$ is no integer.

Since I assumed a Fermat Triple to exist,

$\sum_{k=0}^{s}(C^2)^{s-k}\,(B^2)^k$,

must be an integer.

Hence, $\frac{A^n}{(C-B)(C+B)}$, must be an integer.

For a Fermat Triple to exist, $(C+B)$ is an integer factor of $\sum_{k=0}^{n-1} C^{n-1-k}\,B^{k}$.

The fundamental theorem of arithmetic.

End of Proof

Since $(\sum_{k=0}^{n-1} C^{n-1-k}\,B^{k})$ is an integer factor of A^n, $(C+B)$ is an integer factor of A^n.

For n is even:

$C^n - B^n = \left(C^{\frac{n}{2}} + B^{\frac{n}{2}}\right)\left(C^{\frac{n}{2}} - B^{\frac{n}{2}}\right) = A^n$.

Now, I use this factorization for:

$n = 2(s+1)$ with $s = 1(1)\ldots$ and I preclude $n = 2^r$ with r= $2(1)\ldots$.

I deal with $n = 2^r$ later on.

Then, **(3.1)** is factorized in the following way:

$C^n - B^n = \left(C^{\frac{n}{l}} + B^{\frac{n}{l}}\right)\left(C^{\frac{n}{l}} - B^{\frac{n}{l}}\right) = \left(C^{\frac{n}{l}} + B^{\frac{n}{l}}\right)\ldots\ldots\ldots\left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)\left(C^{\frac{n}{q}} - B^{\frac{n}{q}}\right) =$

$= \left(C^{\frac{n}{l}} + B^{\frac{n}{l}}\right)\ldots\ldots\ldots\left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)(C-B)\left(\sum_{p=0}^{\frac{n}{q}-1} C^{\frac{n}{q}-1-p}\,B^{p}\right) = A^n$, **(3.17)**

with $l = 2(2)\ldots.q$, and

$\left(C^{\frac{n}{q}} - B^{\frac{n}{q}}\right) = (C - B)\left(\sum_{p=0}^{\frac{n}{q}-1} C^{\frac{n}{q}-1-p}\, B^{p}\right)$.

Note: for $^{n}/_{q}$ even, factorizing can be continued.

The factorizing of $(C^{n} - B^{n})$ is ended for the first value of $^{n}/_{q}$ odd .

To demonstrate **(3.17)**, choose $n = 12$:

$C^{12} - B^{12} = A^{12} \rightarrow (\,C^{6} + B^{6})\,(C^{6} - B^{6}) = A^{12} \rightarrow$

$\rightarrow (C^{6} + B^{6})(C^{3} + B^{3})(C^{3} - B^{3}) = (C^{6} + B^{6})(C^{3} + B^{3})(C - B)\left(\sum_{p=0}^{2} C^{2-p}\, B^{p}\right) = A^{12}.$

$\left(C^{\frac{n}{l}} + B^{\frac{n}{l}}\right)$, ..., and $\left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)$ are all odd and factors of A^{n}**.**

Proof

Let $\left(C^{\frac{n}{l}} + B^{\frac{n}{l}}\right) \cdot \ldots \cdot \left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)$ not be a factor of A^{n}.

Consequently $\frac{A^{n}}{\left(C^{\frac{n}{l}}+B^{\frac{n}{l}}\right)\cdot\ldots\cdot\left(C^{\frac{n}{q}}+B^{\frac{n}{q}}\right)}$ is not an integer factor.

Then, $(C - B)\left(\sum_{p=0}^{\frac{n}{q}-1} C^{\frac{n}{q}-1-p}\, B^{p}\right)$ cannot be an integer.

For a Fermat Triple to exist $\left(C^{\frac{n}{l}} + B^{\frac{n}{l}}\right) \cdot \ldots \cdot \left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)$ has to be a factor of A^{n} and has to be an integer.

Consequently, the assumption $\left(C^{\frac{n}{l}} + B^{\frac{n}{l}}\right) \cdot \ldots \cdot \left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)$ be no integer factor of A^{n} is contradicted.

Hence, $\left(C^{\frac{n}{l}} + B^{\frac{n}{l}}\right) \cdot \ldots \cdot \left(C^{\frac{n}{q}} + B^{\frac{n}{q}}\right)$, the summation in **(3.2)**, is an integer factor of A^{n}.

To summarize: it is about the application of *The fundamental theorem of arithmetic.*

End of proof.

The remaining factor, $A_{4} \equiv \left(C^{\frac{n}{q}} - B^{\frac{n}{q}}\right) = \;(C - B)\left(\sum_{p=0}^{\frac{n}{q}-1} C^{\frac{n}{q}-1-p}\, B^{p}\right)$.

Use the analysis for $\boldsymbol{n}$ **is odd** of section 3.3:

$(C - B)\left(\sum_{p=0}^{\frac{n}{q}-1} C^{\frac{n}{q}-1-p}\, B^{p}\right) = C^{\frac{n}{q}} - B^{\frac{n}{q}}.$ **(3.18)**

$C^{\frac{n}{q}} - B^{\frac{n}{q}}$, is a factor of A^{n}. **(3.19)**.

$^{n}/_{q}$ is odd and can be expanded in a product of powers of prime numbers. The analysis of section 3.3 can be used.

Consequently, B is irrational and

there are no Fermat Triples for $\boldsymbol{n}$ is even, $n = 2(s + 1)$ **with** $s = 1(1)$ ... **and** $n = 2^{r}$ **with** $r = 2(1)$ ... precluded.

3.6 Fermat Triples with n =2^{r}

$\{\exists [A, B, C] \in \mathbb{P}\}$.

To confirm this assumption, the question to be answered is: can I obtain integer roots for B

from **(3.6)** for any positive integer value of $(C-B)=A_1$?
$\{A,C\in\mathbb{P}|A$ and C odd$\}$, $\{B\in\mathbb{P}|B$ even$\}$.

Now, $n=2^r$ for $r=2(1)$ .....
I start with $n=4$.
With **(3.5)**:
$B^3+\sum_{k=2}^{n-1}\frac{1}{4}\binom{4}{k}A_1^{k-1}B^{4-k}=\frac{1}{4A_1}(A^4-A_1^4)$.
Is $\{B\in\mathbb{N}|B$ even$\}$?

Proof
$B^3+\frac{3}{2}A_1B^2+A_1^2B=\frac{1}{4A_1}(A-A_1)(A+A_1)(A^2+A_1^2)$.
Now,
$\{(A\pm A_1)\in\mathbb{P}|$even$\}$.
Then,
$\frac{1}{4A_1}(A-A_1)(A+A_1)(A^2+A_1^2)=\frac{A-A_1}{2}\frac{A+A_1}{2}(\frac{A^2}{A_1}+A_1)$.
Use **(2.2)** and **(3.3)**:
$\frac{A^2}{A_1}=\frac{P_l^2}{P_j^2}$ →no integer.
Consequently, $\frac{A-A_1}{2}\frac{A+A_1}{2}(\frac{A^2}{A_1}+A_1)$ is no integer.
With:
$\{B\in\mathbb{P}|B$ even$\}$,
$B^3+\frac{3}{2}A_1B^2+A_1^2B$ is an integer.
Hence,
The left-hand side of
$B^3+\frac{3}{2}A_1B^2+A_1^2B=\frac{1}{4A_1}(A-A_1)(A+A_1)(A^2+A_1^2)$,
is an integer and the right-hand side is not.
So, B cannot be an integer.
A contradiction.
No Fermat Triples for $n=4$.
End of Proof.

For $n=4$, with (**3.2)**
$(C-B)(C+B)(C^2+B^2)=A^4$. **(3.20)**
$(C-B), C+B)$ and (C^2+B^2) are odd integer factors of A^4, with no factors in common.

Proof
In section 3.1, I proved $(C-B)\equiv A_1$ and $(\sum_{k=0}^{3}C^{3-k}B^k)$ to be integer factors of A^4 for a Fermat Triple to exist.
Now, $(\sum_{k=0}^{3}C^{3-k}B^k)$ can be factorized as $(C+B)(C^2+B^2)$.
Consequently, for a Fermat Triple to exist, $(C+B)$ and (C^2+B^2) are integer factors of A^4.
Straightforward application of *The fundamental theorem of arithmetic.*
End of Proof.

In addition, for $n = 4$,
with **(3.2):** $(C-B)(C+B)(C^2+B^2) = A^4$,
the proof of the nonexistence of integer solutions has been given with the so-called method of 'Descente infinie' (Giorello, G., et al).
There is **no Fermat triple for $n = 4$.**

What about $n > 4$, and , $n = 2^r, \{r \in \mathbb{P} | r > 2\}$?
(3.2) can be written as:
$(C-B)(\sum_{k=0}^{n-1} C^{n-1-k} B^k) = (C-B) \prod_{k=0}^{r-1} (C^{2^k} + B^{2^k}) = A^n$,
or
$$(C-B)(C+B)(C^2+B^2) \ldots\ldots\ldots (C^{2^{r-1}} + B^{2^{r-1}}) = A^n. \quad \textbf{(3.21)}$$
(3.21):
Proof by induction
Assume **(3.21)** to be correct for $(r-1)$. What about r? Set $A^n = C^{2^r} - B^{2^r}$ in **(3.21)**.
Multiply **(3.21)** to the left and the right with $(C^{2^r} + B^{2^r}) \rightarrow$
$(C-B)(C+B)(C^2+B^2) \ldots\ldots\ldots (C^{2^{r-1}} + B^{2^{r-1}})(C^{2^r} + B^{2^r}) = (C^{2^r} - B^{2^r})(C^{2^r} + B^{2^r})$,
$\rightarrow (C-B)(C+B)(C^2+B^2) \ldots\ldots\ldots (C^{2^r} + B^{2^r}) = (C^{2^{r+1}} - B^{2^{r+1}})$.
The fundamental theorem of arithmetic.
End of proof.

In section 3.2 on *Factorizing of* $C^n - B^n = A^n$, I proved $(C-B)$ and $(\sum_{k=0}^{n-1} C^{n-1-k} B^k)$ to be integer factors of A^n with the assumption $A, B, C,$ to be a Fermat triple.
The expressions in **(3.21)** , $(C^{2^k} + B^{2^k})$, are integer factors of A^n, with no factors in common, for a Fermat triple to exist.
Then,
$(C-B)(C+B)(C^2+B^2)$ is an integer factor of A^n.
Hence, the analysis for $n = 4$ leads to the conclusion B to be an Algebraic irrational number.
This analysis still applies for $n > 4$. I factorize A^n and equate an integer factor of A^n with $(C-B)(C+B)(C^2+B^2) \rightarrow B$ an Algebraic irrational number.
So,
no Fermat Triple for $n > 4$, and , $n = 2^r, \{r \in \mathbb{P} | r > 2\}$.

I can use the results for $n = 4$ in another way, with $n = 2^r$ and $r = 3(1) \ldots$:
(3.2) $\rightarrow \left(C^{2^{(r-2)}}\right)^4 - \left(B^{2^{(r-2)}}\right)^4 = \left(A^{2^{(r-2)}}\right)^4$.
Hence, for $(A^{2^{(r-2)}})$ assumed to be an integer, $(C^{2^{(r-2)}})$ and $(B^{2^{(r-2)}})$ are no integers.
See section 3.1.1
Note: when $(B^{2^{(r-2)}})$ is assumed to be rational number and expressed as a quotient of (powers of) prime numbers, P_m and P_h respectively, with no common factors:
$(B^{2^{(r-2)}}) = \frac{P_m}{P_h} \rightarrow B = (\frac{P_m}{P_h})^{\frac{1}{2^{(r-2)}}}$,
the analysis of section 3.3 can be used.
This analysis leads to the conclusion:

B is not an integer and **there are no Fermat Triples for $n > 4$.**

Hence , for any positive, integer value of $A_1 (= C - B)$:
there are no Fermat Triples for $n = 2^r$ **with** r$= 2(1)$ … .
Consequently,
There is no Fermat triple for n is even; $n = 2(r+1)$ **,** $\{r \in \mathbb{P}\}$.

§4. Conclusions.

The nonexistence of a Fermat Triple has been proven in §3 for a given value of the odd integer A and any positive integer value of $(C - B)$, odd and $0 < (C - B) < A$.

- In § 3.1 the non-existence of Fermat Triples have been proven for $n = m \cdot 3$, and $n = m \cdot 4$,
  using the results for $n = 3$, and $n = 4$.
- In § 3.3 has been proven B not to be an integer for n is prime.
  The method of Ascent Infinite is introduced.
  n is assumed to be an integer factor of A.
- In § 3.4 for n is a product of powers of odd prime and even numbers, B is an Algebraic irrational number.
  n is assumed to be an integer factor of A.
- In § 3.5 for n is even and a product of prime numbers, B is an Algebraic irrational number. In addition, the existence of Pythagorean triples has been proven.
- In § 3.6 for $n = 2^r$, $\{r \in \mathbb{P} | r \geq 2\}$, B is an Algebraic irrational number.

On basis of these results, I conclude **there exist no Fermat Triples ($n \geq 3$).**
The key to the solution is: **n is prime.**

For further reading on de Fermat's last theorem I like to mention Simon Singh's book. Finally, to conclude the above approach on de Fermat's last theorem I like to cite Feynman on de Fermat's last theorem: "*For my money Fermat's theorem is true*". Feynman estimated that the probability of finding integer solutions is less than 10^{-200} (Schweber).

§5. Literature:

Giorello, G. and C. Sinigaglia, *Fermat- I sogni di un magistrato all'origine della matematica moderna,* 2001, Le Science, Milano.

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