De Fermat’s Last Theorem and the margin of De Fermat’s Diophantus

The emergence of Format’s Last Theorem from Pythagorean Triples
Update: 2025-11-26: §4 edited

Leen Noordzij

Dr.l.noordzij@leennoordzij.nl
www.leennoordzij.me

Content

Abstract

Another proof of De Fermat’s last theorem is presented.
A concise proof of De Fermat’s last theorem is obtained by transforming de Fermat’s equation into a Pythagorean Equation. By comparing the resulting Pythagorean Triples with the factorised De Fermat’s equation, De Fermat’s Last theorem is proved with help of the proof by contradiction.

Tags: Culture, Descente Infinie, Diophantus, Education, De Fermat, Fundamental, Ground State Triples, Theorem of Arithmetic, History, Mathematics, Prime Numbers, Proof by Contradiction, Pythagorean Triples, Andrew Wiles.

§1. Introduction

‘*Fermat’s equation:* $a^n + b^n = c^n$, **(1.1)**

has no solutions $\{a, b, c \text{ and } n \in \mathbb{N} | n \geq 3\}$
A statement by Andrew Wiles written on the black board after the presentation of the conclusions concerning equation **(1.1)**, Giorello, G. and C. Sinigaglia. Wiles received the *Fields Medal* for the proof of De Fermat’s Last Theorem.

Is there another proof of the last theorem of de Fermat, fitting into the margin of de Fermat's copy of Diophantus?
De Fermat mentioned his proof not to fit in the width of the margin of his Diophantus copy. An Ockham's razor, sort of. This stimulated me to investigate De Fermat's last theorem. Which tools could have been used by De Fermat?
Well, he could have used the analysis of Pythagoras and Euclid. In addition, we used the proof by contradiction: reductio ad absurdum. The form of argument that attempts to establish a claim by showing that the opposite scenario would lead to absurdity or contradiction,
https://en.wikipedia.org

§2 The elements of the Pythagorean Triples

In this paragraph $n = 2$ in **(1.1)** will be analysed. Solutions of b and c are obtained for a given value of the natural number a, the so-called Pythagorean Triples. See also Noordzij(1).
I assume the existence of a solution of **(1.1)**, $n = 2$, denominated a "Pythagorean Triple": $[\, a, b, c]$, $\{b \in \mathbb{N} | b > 0$ and even$\}$ and $\{a, c \in \mathbb{N} | a, c > 0$ and odd$\}$.
Next, we factorize $a = a_1 a_2$, where a_1, a_2 are products of powers of prime numbers and no common factors[1]. Here, I assume a to be odd as mentioned above and $a_1 > a_2$.
Obviously, with
$a = a_1 \rightarrow a_2 = 1$, a "ground state" of Pythagorean Triples for a particular a number is obtained.
With **(1.1)**, $n = 2$:

$$c = \sqrt{(a_1 a_2)^2 + b^2}\,. \qquad \textbf{(2.1)}$$

If I can compose an expression for b^2 of which $(a_1 a_2)^2$ in **(2.1)** is part of a cross product of b^2, then an integer c can be obtained., a relation between a and b has to be constructed. See Noordzij(2). Below a brief summary.

$$b^2 = \frac{1}{4}(a_1^4 - 2(a_1 a_2)^2 + a_2^4) \rightarrow b = \frac{1}{2}(a_1^2 - a_2^2)\,.$$

Substitute $b^2 = \frac{1}{4}[a_1^4 - 2(a_1 a_2)^2 + a_2^4]$ into **(2.1)**
Hence, c in **(2.1)** :

$$c = \sqrt{(a_1 a_2)^2 + b^2} = \sqrt{\frac{1}{4}(a_1^4 + 2(a_1 a_2)^2 + a_2^4)} = \frac{1}{2}(a_1^2 + a_2^2). \qquad \textbf{(2.2)}$$

So,
with $a = a_1 a_2$, a natural number, **(2.3)**

$$b = \frac{1}{2}(a_1^2 - a_2^2) = \frac{1}{2}(a_1 + a_2)(a_1 - a_2), \qquad \textbf{(2.4)}$$

$$c = \frac{1}{2}(a_1^2 + a_2^2) = \frac{1}{2}(a_1 + a_2)^2 - a, \qquad \textbf{(2.5)}$$

are both natural numbers and $[a, b, c]$ represents a Pythagorean Triple.
In addition, as shown by **(2.4)** and **(2.5)**, b is even and c is odd.

[1] With common factors no primitive triple is obtained.

These equations play a key role in the proof of De Fermat's Last Theorem.
In §2 in Noordzij(4), more Pythagorean triples are discussed and two examples of De Fermat's Last Theorem are presented.

§3 From Pythagoras and Back to De Fermat's Last Theorem, the Hybrid Equation, and the Table of Contradictions

See also *De Fermat's Last Theorem, Pythagorean Triples and The Descente Infinie revisited*, www.leennoordzij.me , Noordzij (1). There, in §7, the question is raised whether the proof presented is sufficient. Here we discus the proof in more detail.

For a given natural number a, and $\{a, b, c \in \mathbb{N} | > 1, a$ and c odd, b even$\}$, $\{n \in \mathbb{N} | n > 2\}$, we presume a De Fermat's Triple to exists.
There are no triples for randomly chosen values of the integers a and b Noordzij(3). A relation between a, b and c in **(1.1)** is necessary and sufficient? For this relation, use will be made of factorization of De Fermat's equation and the existence of Pythagorean Triples.
Factorization of $c^n - b^n$:
$a^n + b^n = c^n \to c^n - b^n = a^n \to$
$\to (c-b)(\sum_{k=0}^{n-1} c^{n-1-k}\, b^k) = a^n.$ **(3.1)**

Proof   of $(c-b)(\sum_{k=0}^{n-1} c^{n-1-k}\, b^k) = c^n - b^n$: $(c-b)(\sum_{k=0}^{n-1} c^{n-1-k}\, b^k) = \sum_{k=0}^{n-1} c^{n-k}\, b^k - \sum_{k=0}^{n-1} c^{n-1-k}\, b^{k+1} \to$ $\to c^n + \sum_{k=1}^{n-1} c^{n-k}\, b^k - b^n - \sum_{k=0}^{n-2} c^{n-1-k}\, b^{k+1}$.   Now change the summation index of $\sum_{k=1}^{n-1} c^{n-k}\, b^k$, into $k-1=r$ and we have: $c^n - b^n + \sum_{r=0}^{n-2} c^{n-1-r}\, b^{r+1} - \sum_{k=0}^{n-2} c^{n-1-k}\, b^{k+1}$.   Hence $\sum_{r=0}^{n-2} c^{n-1-r}\, b^{r+1} - \sum_{k=0}^{n-2} c^{n-1-k}\, b^{k+1} = 0$.   Consequently $(c-b)(\sum_{k=0}^{n-1} c^{n-1-k}\, b^k) = c^n - b^n$.   ***End of Proof.***

With the presumption of a, b and c being a De Fermat Triple,
$(c-b)$ and $(\sum_{k=0}^{n-1} c^{n-1-k}\, b^k)$
are natural numbers. Both $(c-b)$ and $(\sum_{k=0}^{n-1} c^{n-1-k}\, b^k)$ are integer factors of a^n, consisting of products of powers of prime numbers.

Proof:
Let $(c-b)$ be no integer factor of a^n.
Consequently $\frac{a^n}{(c-b)}$ is not an integer.
Then
$\sum_{k=0}^{n-1} c^{n-1-k}\, b^k)$ cannot be an integer.
For a Fermat triple $(c-b)$ has to be a factor of a^n and has to be an integer.
Hence,
$\sum_{k=0}^{n-1} c^{n-1-k}\, b^k$, is an integer factor of a^n.
End of proof.

So, $(c-b)$ to be a natural number is a condition to find De Fermat triples.
Given $a = a_1 \cdot a_2$, and where a_1, a_2 are products of powers of prime numbers and no common factors, we set
$c - b = a_2^n$, and $\sum_{k=0}^{n-1} c^{n-1-k}\, b^k = a_1^n$.

De Fermat's theorem in a nutshell
$a^n + b^n \neq c^n$,
with
$\{a, b, c \in \mathbb{N}|$ and $> 0\}$ and $\{n \in \mathbb{N}|n > 2\}$.

Let us look once more into De Fermat's equation and transform De Fermat's equation into the Pythagorean equation:
$a^n + b^n = c^n \rightarrow (a^{n/2})^2 + (b^{n/2})^2 = (c^{n/2})^2$.
In the following the proof of De Fermat's Last Theorem is presented.
Use is made of proof by contradiction.

Proof
$(a^{n/2})^2 + (b^{n/2})^2 = (c^{n/2})^2$, **(3.2)**
represents a hybrid equation:
- De Fermat's equation, where $[a, b, c]$ is presumed to represent a De Fermat Triple,
- the Pythagorean equation, where $[a^{\frac{n}{2}}, b^{\frac{n}{2}}, c^{\frac{n}{2}}]$ is presumed to constitute a Pythagorean Triple for any integer $n \geq 3$ [2].

The questions to be answered for a given natural number $a > 1$
- can b and c be natural numbers, for $n > 2$?
- do the Pythagorean Triples $[a^{\frac{n}{2}}, b^{\frac{n}{2}}, c^{\frac{n}{2}}]$ exist?

We presume $\{a, b, c \in \mathbb{N}|$ and $> 1\}$ and $\{n \in \mathbb{N}|n > 2\}$,
- $(a^{n/2})^2 + (b^{n/2})^2 = (c^{n/2})^2$, (Proof continued on next page)
- the Pythagorean Triple in **(3.2)** $[a^{\frac{n}{2}}, b^{\frac{n}{2}}, c^{\frac{n}{2}}]$ exists,
- the De Fermat's Triple $[a, b, c]$ exists.

---

[2] For $n = 3$, De Fermat showed the non-existence of so-called De Fermat's Triples using the method of *Descente Infinie,* Noordzij (1). Caveat Tres: $a^{\frac{n}{2}}$ can still be a natural number for a being a quadratic number and/or n an even number.

A claim to be tested.
We use the results from the proof of the existence of Pythagorean triples in §2, eqs. **(2.4)** and **(2.5)** with $a^{n/2} = a_1^{n/2} \cdot a_2^{n/2}$:
$a_2^n = c^{\frac{n}{2}} - b^{\frac{n}{2}}$, and $a_1^n = c^{\frac{n}{2}} + b^{\frac{n}{2}}$.
Using **(3.1)**
$a_2^n = c - b$, and $a_1^n = \sum_{k=0}^{n-1} c^{n-1-k}\, b^k$
Now, we have obtained two sets of contradicting expressions[3] :
$a_2^n = c^{\frac{n}{2}} - b^{\frac{n}{2}}, a_1^n = c^{\frac{n}{2}} + b^{\frac{n}{2}}$
and
$a_2^n = c - b\,, a_1^n = \sum_{k=0}^{n-1} c^{n-1-k}\, b^k$.
In the table below we present the consequences:
Table of Contradictions

Suppose $a_2^n = c^{\frac{n}{2}} - b^{\frac{n}{2}}$, to be true, then	Suppose $a_2^n = c - b$, to be true, then
$a_2^n = c - b$ cannot be true $\therefore$ $c - b \neq a_2^n$, $\rightarrow b$ and/or c cannot be natural numbers. With $c - b \neq a_2^n$ $(c-b)(\sum_{k=0}^{n-1} c^{n-1-k}\, b^k) \neq a_2^n \cdot a_1^n = a^n \rightarrow$ $\rightarrow c^n - b^n \neq a^n$.	$a_2^n = c^{\frac{n}{2}} - b^{\frac{n}{2}}$ cannot be true $\therefore$ $c^{\frac{n}{2}} - b^{\frac{n}{2}} \neq a_2^n \rightarrow$ $\rightarrow \left(c^{\frac{n}{2}} + b^{\frac{n}{2}}\right)\left(c^{\frac{n}{2}} - b^{\frac{n}{2}}\right) \neq a_1^n \cdot a_2^n = a^n \rightarrow$ $\rightarrow (a^{n/2})^2 + (b^{n/2})^2 \neq \left(c^{n/2}\right)^2$.
$\therefore$ $a^n + b^n \neq c^n$, b and/or c cannot be natural numbers. $\therefore$ The triples $[a, b, c]$ in $a^n + b^n = c^n$, do not exist as has been presumed. The last theorem of De Fermat is confirmed.	Consequently the triples $[a^{\frac{n}{2}}, b^{\frac{n}{2}}, c^{\frac{n}{2}}]$ in $(a^{n/2})^2 + (b^{n/2})^2 = \left(c^{n/2}\right)^2$ do not exist. $b^{\frac{n}{2}}$ and/or $c^{\frac{n}{2}}$ cannot represent natural numbers. From this we cannot conclude b and/or c not to be natural numbers. With, e.g. $c^{n/2}$ not to be a natural number: $c^{n/2} = 17\sqrt{17}$, say. We find with $n = 3$: $c = (17\sqrt{17})^{2/3} = 17$.
We have $a^n + b^n \neq c^n$. Then $\left(a^{\frac{n}{2}}\right)^2 + \left(b^{\frac{n}{2}}\right)^2 \neq \left(c^{\frac{n}{2}}\right)^2$. Hence, the triples $[a^{\frac{n}{2}}, b^{\frac{n}{2}}, c^{\frac{n}{2}}]$ do not exist. Consequently, the presumption $a_2^n = c^{\frac{n}{2}} - b^{\frac{n}{2}}$, is not correct.	With $(a^{n/2})^2 + (b^{n/2})^2 \neq \left(c^{n/2}\right)^2$, $\therefore$ $a^n + b^n \neq c^n$. Consequently, the presumption $a_2^n = c - b$, is not correct and b and/or c cannot be natural numbers. Hence, the triples $[a, b, c]$ do not exist. The last theorem of De Fermat is confirmed.

Conclusion of the proof on the preceding page:
For a given natural number a:
b and c are not natural numbers.

[3] There is no contradiction for $n = 2$, as expected.

To summarize
De Fermat's Last Theorem shows itself using Pythagorean Triples and factorization of De Fermat's equation.
De Fermat's Last Theorem is correct:
the claim $\{a, b, c \in \mathbb{N} |$ and $> 1\}$ and $a^n + b^n = c^n$ for $n > 2$ is assessed and rejected using reductio ad absurdum.
End of Proof

§4 Discussion and Conclusions

The analysis of §3 applies for a is odd or even.
In Noordzij (2) triples for a is odd or even are derived.
In Noordzij (2), we also presented a different set of formulas for Pythagorean Triples: formulas **(1.12)** and **(1.13)** . These are based on the factorization of
$a^2 = a_3 a_4$,
with a_3 and a_4 being a product of powers of prime numbers with no common factors
The formulas **(1.12)** and **(1.13)** in Noordzij (2)[4]:
$$b = \frac{1}{2}(a_3 - a_4), \qquad \textbf{(4.1)}$$
and
$$c = \frac{1}{2}(a_3 + a_4). \qquad \textbf{(4.2)}$$
We use these expressions similar to the analysis of §3.
Then
$$a^n = a_3^{n/2} \cdot a_4^{n/2}.$$
Derived from the expressions for the Pythagorean Triples **(4.1)** and **(4.2)**
$$b^{n/2} = \frac{1}{2}\left(a_3^{n/2} - a_4^{n/2}\right),$$
$$c^{n/2} = \frac{1}{2}\left(a_3^{n/2} + a_4^{n/2}\right).$$
With these expressions we obtain:
$$c^{n/2} - b^{n/2} = a_4^{n/2} \rightarrow a_4^{n/2} = c^{n/2} - b^{n/2}.$$
Using **(3.1)**
$$a_4^{n/2} = c - b.$$
Again we obtain two contradicting expressions:
$$a_4^{n/2} = c^{n/2} - b^{n/2},$$
and
$$a_4^{n/2} = c - b.$$
With help of the Table of Contradictions in §3, the following conclusions are found:
De Fermat's Theorem is found to be correct using the approach via the route of Pythagorean Triples. Or: "*De Fermat's Last Theorem hidden in Pythagorean Triples*".

[4] For a is even to obtain primitive triples a^2 needs to be factorised. With a is odd a or a^2 can be factorised.

The expressions used for the elements of the Pythagorean Triples are derived in §2, expressions (**2.4**) and (**2.5**), with $a_3 = a_1^2$ and $a_4 = a_2^2$.
In §3 we transformed De Fermat's equation into the Pythagorean equations and finally obtained the hybrid equation
$(a^{n/2})^2 + (b^{n/2})^2 \neq (c^{n/2})^2 \rightarrow a^n + b^n \neq c^n$,
proving De Fermat's Last Theorem to be correct.
Whether or not this paper represents the tools used by De Fermat remains unknown, alas.

§5 Literature

Giorello, G. and C. Sinigaglia, Fermat- I sogni di un magistrato all'origine della matematica moderna, 2001, Le Science, Milano.

Noordzij, L. (1), *De Fermat's Last Theorem, Pythagorean Triples and The Descente Infinie revisited,* www.leennoordzij.me .

Noordzij, L. (2), *Pythagorean Triples, Goldbach's Conjecture and De Fermat's Last Theorem,* www.leennoordzij.me .

Noordzij, L (3), *Fermat's Last Theorem and Pythagoras*, www.leennoordzij.me .

Noordzij, L (4), *From De Fermat to Pythagoras and Back.*
The emergence of Format's Last Theorem from Pythagorean Triples, www.leennoordzij.me .

The Economist, *Ancient Geometry, No need for a protractor*, Section Science and Technology, August 7th 2021.